Solution (#834) Denote the first determinant as D_n. Then expanding down the first column we have

$$\begin{aligned} D_n &= (\alpha+\beta)\begin{vmatrix} \alpha+\beta & \alpha & 0 & \cdots & 0 \\ \beta & \alpha+\beta & \alpha & \ddots & \vdots \\ 0 & \beta & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha+\beta & \alpha \\ 0 & \cdots & 0 & \beta & \alpha+\beta \end{vmatrix} - \beta \begin{vmatrix} \alpha & 0 & 0 & \cdots & 0 \\ \beta & \alpha+\beta & \alpha & \ddots & \vdots \\ 0 & \beta & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha+\beta & \alpha \\ 0 & \cdots & 0 & \beta & \alpha+\beta \end{vmatrix} \\ &= (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2}, \end{aligned}$$

So we have

$$D_n - (\alpha+\beta)\, D_{n-1} + \alpha\beta D_{n-2} = 0.$$

From our knowledge of recurrence relations we have that

$$D_n = A\alpha^n + B\beta^n$$

for some constants A, B. Now by calculation and inspection

$$D_1 = \alpha+\beta = \frac{\alpha^2-\beta^2}{\alpha-\beta} \quad \text{and} \quad D_2 = \alpha^2+\alpha\beta+\beta^2 = \frac{\alpha^3-\beta^3}{\alpha-\beta}.$$

Hence $A = -B = (\alpha-\beta)^{-1}$ and we have

$$D_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}.$$

If Δ_n denotes the second given determinant then we can down the first column to again arrive at

$$\Delta_n = b\Delta_{n-1} - ca\Delta_{n-2}.$$

As with the first determinant we then have that

$$\Delta_n = A\gamma^n + B\delta^n$$

for some A, B and where γ, δ are the roots of

$$x^2 - bx + ca = 0.$$

As $\Delta_1 = b$ and $\Delta_2 = b^2 - ac$ we have

$$b = A\gamma + B\delta, \qquad b^2 - ac = A\gamma^2 + B\delta^2.$$

With some algebra, and noting $b = \gamma+\delta$ and $ac = \gamma\delta$, we see that

$$A = \frac{\gamma}{\gamma-\delta}, \qquad B = \frac{\delta}{\delta-\gamma}.$$

Thus

$$\Delta_n = \frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}.$$

If we wish to rewrite this in terms of a, b, c we can write

$$\gamma = \frac{b+\sqrt{b^2-4ac}}{2}, \qquad \delta = \frac{b-\sqrt{b^2-4ac}}{2}$$

and then

$$\begin{aligned} \Delta_n &= \frac{\left(b+\sqrt{b^2-4ac}\right)^{n+1} - \left(b-\sqrt{b^2-4ac}\right)^{n+1}}{2^{n+1}\sqrt{b^2-4ac}} \\ &= 2^{-n}\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2l+1} b^{n-2l}(b^2-4ac)^l, \end{aligned}$$

by the binomial theorem.