Solution (#856) For an $n \times n$ matrix $B = (b_{ij})$ then (3.51) states that

$$\det B = b_{1i}C_{1i}(B) + b_{2i}C_{2i}(B) + \cdots + b_{ni}C_{ni}(B)$$

and Definition 3.146 states that for an $n \times n$ matrix $A = (a_{ij})$

$$\det A = a_{11}C_{11}(A) + a_{21}C_{21}(A) + \cdots + a_{n1}C_{n1}(A).$$

So set $B = AS_{i1}$ and applying Definition 3.146 to B we get

$$\det(B) = b_{11}C_{11}(B) + b_{21}C_{21}(B) + \cdots + b_{n1}C_{n1}(B).$$

Now $B = AS_{i1}$ is the matrix A with the ith and first columns swapped. Hence $b_{k1} = a_{ki}$. We also know by the product rule for determinants that

$$\det B = \det A \det S_{i1} = -\det A.$$

So we have

$$-\det(A) = a_{1i}C_{11}(B) + a_{2i}C_{21}(B) + \cdots + a_{ni}C_{n1}(B).$$

Now if the columns of A are $\mathbf{c}_1, \cdots, \mathbf{c}_n$, and we use $\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n$ to denote these columns with their kth entries removed, then we have

$$\begin{aligned}
C_{k1}(B) &= (-1)^{k+1} \det(\tilde{\mathbf{c}}_2 | \cdots | \tilde{\mathbf{c}}_{i-1} | \tilde{\mathbf{c}}_1 | \tilde{\mathbf{c}}_{i+1} | \cdots | \tilde{\mathbf{c}}_n) \\
&= (-1)^{k+1} (-1)^{i-2} \det(\tilde{\mathbf{c}}_1 | \cdots | \tilde{\mathbf{c}}_{i-1} | \tilde{\mathbf{c}}_{i+1} | \cdots | \tilde{\mathbf{c}}_n) \qquad [\text{moving the } i\text{th column past } i-2 \text{ columns}] \\
&= -(-1)^{k+i} \det(\tilde{\mathbf{c}}_1 | \cdots | \tilde{\mathbf{c}}_{i-1} | \tilde{\mathbf{c}}_{i+1} | \cdots | \tilde{\mathbf{c}}_n) \\
&= -\det C_{ki}(A).
\end{aligned}$$

Hence we have

$$-\det(A) = -a_{1i}C_{1i}(A) - a_{2i}C_{2i}(A) - \cdots - a_{ni}C_{ni}(A)$$

and the desired result follows.