Solution (#863) Let P be an $n \times n$ permutation matrix expressible as

$$P = R^{-1}\mathrm{diag}(\Sigma_{r_1}, \Sigma_{r_2}, \ldots, \Sigma_{r_k})R$$

where R is a permutation matrix. Note that

$$\begin{aligned} P^N = I_n \qquad &\iff \qquad \mathrm{diag}(\Sigma_{r_1}, \Sigma_{r_2}, \ldots, \Sigma_{r_k})^N = I_n \\ &\iff \qquad (\Sigma_{r_i})^N = I_{r_i} \quad \text{for each } i. \end{aligned}$$

Further we can note that

$$(\Sigma_r)^r = I_r$$

and that this is the first positive power of Σ_r which returns to the identity for

$$(\Sigma_r)^k \mathbf{e}_1^T = \mathbf{e}_{k+1}^T \neq \mathbf{e}_1^T \quad \text{for} \quad 1 \leqslant k < r.$$

More generally we will have $(\Sigma_r)^s = I_r$ if and only if s is a multiple of r. Hence the smallest positive integer N such that $(\Sigma_{r_i})^N = I_{r_i}$ for each i equals the smallest positive integer N which is a multiple of each r_i, their so-called *least common multiple*

$$N = \mathrm{lcm}(r_1, r_2, \ldots, r_k).$$

It again follows that $P^M = I_n$ for every multiple M of N. As $r_k \leqslant n$ then each r_k divides $n!$ and in particular $n!$ is a multiple of N. Hence

$$P^{n!} = I_n$$

as required.