Solution (#873) (i) Suppose that A is singular. In #871 we showed that

$$\det(\text{adj}B) = (\det B)^{n-1}$$

for an invertible matrix B. So we have that

$$\det(\text{adj}(A - xI)) = (\det(A - xI))^{n-1}$$

except for finitely many values of x when $A - xI$ is singular. The LHS and RHS above are then two polynomials that agree except possibly for finitely many values. As two distinct polynomials in x can only agree at most finitely many values it follows that the two polynomials above are equal for all values of x.

The desired result follows when we set $x = 0$.

(ii) Let A and B be $n \times n$ matrices which are not necessarily invertible. Each of the matrices $A - xI_n$ and $B - xI_n$ are invertible except for only finitely many values of x. By #872, except potentially for finitely many values of x, we have

$$\text{adj}((A - xI_n)(B - xI_n)) = \text{adj}(B - xI_n)\text{adj}(A - xI_n).$$

Both sides of this equation are polynomials, which agree except for finitely many values. However two distinct polynomials can only agree on a finite set of roots. It follows that the two sides are in fact the same polynomial. If we set $x = 0$ then we conclude

$$\text{adj}(AB) = \text{adj}B\text{adj}A$$

holds even when A and/or B is singular.