Solution (#883) (i) Consider a linear system, $(A\,|\,\mathbf{b})$ where A is $n\times n$ where $A = LU$. For the forward substitution to solve $L\mathbf{y} = \mathbf{b}$ we have

$$y_1 = b_1; \qquad y_2 = b_2 - l_{21}y_1; \qquad y_3 = b_3 - l_{31}y_1 - l_{32}y_2; \qquad \ldots.$$

So we see that to calculate y_k we need to use $k-1$ subtractions and $k-1$ multiplications. Thus the number of arithmetic operations for this stage equals

$$\sum_{k=1}^{n} 2(k-1) = n(n-1).$$

Having determined $\mathbf{y}$ we then use backward substitution

$$x_n = \frac{y_n}{u_{nn}}; \qquad x_{n-1} = \frac{y_{n-1} - u_{(n-1)n}x_n}{u_{(n-1)(n-1)}}; \qquad x_{n-2} = \frac{y_{n-2} - u_{(n-2)(n-1)}x_{n-1} - u_{(n-2)n}x_n}{u_{(n-2)(n-2)}}; \qquad \cdots$$

which, to calculate x_{n-k}, involves one division, k multiplications and k subtractions, or $2k+1$ operations. Thus the number of arithmetic operations for the backward substitution equals $\sum_{k=0}^{n-1} 2k+1 = n + n(n-1) = n^2$.

Hence, when the LU decomposition is known, the number of operations to determine the solution is $2n^2 - n$. This is the advantage of the LU decomposition: if we need to solve a system $(A\,|\,\mathbf{b})$ for multiple choices of $\mathbf{b}$ and the same A then $2n^2 - n$ is a substantial improvement on row-reducing $(A\,|\,\mathbf{b})$ in each case – see #679 – where we see $\frac{1}{2}n^2(2n-1)$ operations are needed.

(ii) Suppose that an $n\times n$ matrix A has an LU decomposition. We can argue along similar lines to #679 to determine this number. However note that we are not completely row-reducing A to get U but only putting A into echelon form. Note also that as we determine what EROs to perform we determine the entries of L (as seen in Example 278). In this case we are also assuming that $A = LU$ so that no permuting of rows is necessary to begin.

To clear out the first column we need to set

$$a_{ij} \mapsto a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j} \qquad \text{for } 2 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n.$$

This involves working out a multiplier a_{i1}/a_{11} for each row below the first and making then making $n(n-1)$ subtractions and $n(n-1)$ multiplications. So to clear out the first column takes

$$(n-1) + 2n(n-1) = (2n+1)(n-1).$$

operations. For the kth column this number will instead equal

$$(n-k) + 2(n-k+1)(n-k) = (2n-2k+1)(n-k).$$

Hence the total number of operations needed equals

$$\sum_{k=1}^{n-1}(2n-2k+1)(n-k) = \sum_{k=1}^{n-1}(2k+1)k = \frac{1}{6}n(n-1)(4n+1).$$

In particular this number grows as a cubic in n, the leading term being $2n^3/3$.

(iii) Now let L_n denote the number of operations needed to evaluate an $n\times n$ determinant using a Laplace expansion. This involves adding n products together (so $n-1$ additions and n products) where each product involves an $(n-1)\times(n-1)$ determinant. Hence

$$L_n = nL_{n-1} + n + n - 1 = nL_{n-1} + (2n-1).$$

To solve the recurrence relation we can set $M_n = L_n/n!$ so that the relation now reads

$$M_n = M_{n-1} + \frac{2}{(n-1)!} - \frac{1}{n!},$$

which has general solution

$$M_n = A + \sum_{k=1}^{n}\frac{2}{(k-1)!} - \sum_{k=1}^{n}\frac{1}{k!}$$

for a constant A. As $M_1 = 0$ then $A = -1$ and

$$M_n = \left(\sum_{k=0}^{n-1}\frac{1}{k!}\right) - \frac{1}{n!} \qquad \text{and} \qquad L_n = -1 + \sum_{k=0}^{n-1}\frac{n!}{k!}.$$

As n becomes large then M_n approaches e and so the number of operations L_n needed using a Laplace expansion is of the order of $n!e$.