Solution (#890) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$, and α, β be real scalars.

(a) As swapping rows in a determinant changes its sign then

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -\mathbf{v} \wedge \mathbf{u}.$$

(c) As determinants are linear in each row then

$$\begin{aligned} (\alpha\mathbf{u} + \beta\mathbf{v}) \wedge \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (\alpha u_1 + \beta v_1) & (\alpha u_2 + \beta v_2) & (\alpha u_3 + \beta v_3) \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \alpha \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \beta \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \alpha(\mathbf{u} \wedge \mathbf{w}) + \beta(\mathbf{v} \wedge \mathbf{w}). \end{aligned}$$

(d) Say now that $\mathbf{u}, \mathbf{v}$ are perpendicular unit vectors. By #887 we know

$$|\mathbf{u} \wedge \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = 1^2 \times 1^2 - 0^2 = 1$$

and so $|\mathbf{u} \wedge \mathbf{v}| = 1$ and $\mathbf{u} \wedge \mathbf{v}$ is a unit vector.