Solution (#936) Let

$$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\qquad\text{and}\qquad B=\begin{pmatrix}x&1&0&1\\1&x&1&0\\0&1&x&1\\1&0&1&x\end{pmatrix}.$$

(i) Note that expanding down the first column we find

$$c_A(x)=\begin{vmatrix}x&-1&0&0\\0&x&-1&0\\0&0&x&-1\\-1&0&0&x\end{vmatrix}=x(x^3)-(-1)(-1)^3=x^4-1.$$

Hence there are non-real roots of $c_A(x)$ and so the matrix is not diagonalizable over the real numbers. However the four roots of $c_A(x)$ among the complex numbers are distinct and so A is diagonalizable over the complex numbers.

(ii) Note that the columns of A are each of unit length and mutually perpendicular. Hence A is orthogonal and so $A^T=A^{-1}$.

Now $B=A+A^T+xI=A+A^{-1}+xI$. So if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ then $\mathbf{v}$ is an eigenvector of B with eigenvalue $\lambda+\lambda^{-1}+x$. Hence find the eigenvalues of B (or equally the roots of $c_B(x)$) are

$$1+1^{-1}+x=2+x,\qquad -1+(-1)^{-1}+x=-2+x,\qquad i+i^{-1}+x=x,\qquad (-i)+(-i)^{-1}+x=x.$$

As the determinant of an $n\times n$ matrix equals $(-1)^n$ times the product of the roots of the characteristic polynomial (Proposition 3.191) then

$$\det B=(2+x)(-2+x)xx=x^4-4x^2.$$