Solution (#941) (i) For the first matrix, and setting $X = x - a$ for ease of notation, we have

$$\begin{aligned}
\begin{vmatrix} x-a & -b & -c \\ -c & x-a & -b \\ -b & -c & x-a \end{vmatrix} &= \begin{vmatrix} X-b-c & X-b-c & X-b-c \\ -c & X & -b \\ -b & -c & X \end{vmatrix} \\
&= (X-b-c) \begin{vmatrix} 1 & 0 & 0 \\ -c & X+c & -b+c \\ -b & -c+b & X+b \end{vmatrix} \\
&= (X-b-c)\left[X^2 + (b+c)X + (b^2 - bc + c^2)\right].
\end{aligned}$$

The quadratic term has discriminant

$$(b+c)^2 - 4(b^2 - bc + c^2) = -3b^2 + 6bc - 3c^2 = -3(b-c)^2$$

which is typically negative. This means that the characteristic polynomial will not have real roots unless $b = c$.

Suppose then that $b = c$. We then have characteristic polynomial

$$(X - 2b)\left[X^2 + 2bX + b^2\right] = (X - 2b)(X + b)^2$$

and so have eigenvalues $a + 2b, a - b, a - b$.

For $\lambda = a - b$ we consider the null space of the matrix

$$\begin{vmatrix} -b & -b & -b \\ -b & -b & -b \\ -b & -b & -b \end{vmatrix}.$$

As this matrix has rank 1, it has a two dimensional null space and so we will find two independent $(a-b)$-eigenvectors. Provided that $a + 2b \neq a - b$ then we will then be able to find an eigenbasis and our matrix is diagonalizable.

In the case when $a + 2b = a - b$ (so we have three repeated eigenvalues) then $c = b = 0$ and our matrix is aI_3 and so clearly diagonalizable. In conclusion, the matrix is diagonalizable when $b = c$.

(ii) For the second matrix

$$\begin{aligned}
\begin{vmatrix} x-a & -b & -c \\ -a & x-b & -c \\ -a & -b & x-c \end{vmatrix} &= (x-a-b-c) \begin{vmatrix} 1 & -b & -c \\ 1 & x-b & -c \\ 1 & -b & x-c \end{vmatrix} \\
&= (x-a-b-c) \begin{vmatrix} 1 & -b & -c \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} \\
&= x^2(x-a-b-c).
\end{aligned}$$

For $\lambda = 0$ we consider the null space of the matrix

$$\begin{pmatrix} -a & -b & -c \\ -a & -b & -c \\ -a & -b & -c \end{pmatrix}.$$

As this matrix has rank 1, unless $a = b = c = 0$, it has a two dimensional null space and so we will find two independent 0-eigenvectors. Provided that a, b, c are not all zero then we will then be able to find an eigenbasis and our matrix is diagonalizable. In the event that $a = b = c = 0$ then we are looking at 0_{33} which is already diagonal. In conclusion the second matrix is always diagonalizable.