Solution (#945) Let $J(\lambda, r)$ be the $r \times r$ matrix

$$J(\lambda, r) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

As $xI - J(\lambda, r)$ is upper triangular then we can immediately read off (Proposition 3.153) that

$$c_{J(\lambda,r)}(x) = (x - \lambda)^r$$

and so the algebraic multiplicity of λ equals r.

When seeking the λ-eigenvectors, we note that the matrix

$$\lambda I - J(\lambda, r) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

has a null space spanned by $\mathbf{e}_1^T$. So the geometric multiplicity of λ equals 1.

For $k \geqslant 1$ we have

$$(J(\lambda, r) - \lambda I)^k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0_{(r-k)k} & I_{r-k} \\ 0_{kk} & 0_{k(r-k)} \end{pmatrix}.$$

This matrix has rank $r - k$ for $1 \leqslant k < r$ and rank 0 for $k \geqslant r$.