Solution (#962) Say that $|| \quad ||$ is a norm on $\mathbb{R}^n$ and define

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|.$$

M1: By N1 we have $d(\mathbf{x},\mathbf{y}) \geqslant 0$ for all $\mathbf{x}$ and $\mathbf{y}$, and if $d(\mathbf{x},\mathbf{y}) = 0$ then $\mathbf{x}-\mathbf{y} = \mathbf{0}$ by N1 again and $\mathbf{x} = \mathbf{y}$.
M2: By N2 we have

$$d(\mathbf{y},\mathbf{x}) = \|\mathbf{y}-\mathbf{x}\| = \|(-1)(\mathbf{x}-\mathbf{y})\| = |-1|\,\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\| = d(\mathbf{x},\mathbf{y}).$$

M3: Finally for vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ we have

$$\begin{aligned} d(\mathbf{x},\mathbf{z}) &= \|\mathbf{x}-\mathbf{z}\| \\ &= \|(\mathbf{x}-\mathbf{y})+(\mathbf{y}-\mathbf{z})\| \\ &\leqslant \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| \qquad \text{[by N3]} \\ &= d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}). \end{aligned}$$