Solution (#966) For $\mathbf{x}$ in $\mathbb{R}^n$ we define $||\mathbf{x}||_1 = D(\mathbf{0},\mathbf{x})$ and $||\mathbf{x}||_\infty = \delta(\mathbf{0},\mathbf{x})$

We then have

$$||\mathbf{x}||_1 = D(\mathbf{0},\mathbf{x}) = \sum_{k=1}^{n} |x_k|; \qquad ||\mathbf{x}||_\infty = \delta(\mathbf{0},\mathbf{x}) = \max\{|x_1|,|x_2|,\ldots,|x_n|\}.$$

N1: Note that

$$\begin{aligned} ||\mathbf{x}||_1 &\geqslant 0 \quad \text{and} \quad ||\mathbf{x}||_1 = 0 \text{ if and only if } x_i = 0 \text{ for each } i, \text{ i.e. } \mathbf{x} = \mathbf{0}. \\ ||\mathbf{x}||_\infty &\geqslant 0 \quad \text{and} \quad ||\mathbf{x}||_\infty = 0 \text{ if and only if } x_i = 0 \text{ for each } i, \text{ i.e. } \mathbf{x} = \mathbf{0}. \end{aligned}$$

N2: Let $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$\begin{aligned} ||\alpha\mathbf{x}||_1 &= \sum_{k=1}^{n} |\alpha x_k| = |\alpha| \sum_{k=1}^{n} |x_k| = |\alpha|\, ||\mathbf{x}||_1; \\ ||\alpha\mathbf{x}||_\infty &= \max\{|\alpha x_1|,\ldots,|\alpha x_n|\} = |\alpha| \max\{|x_1|,\ldots,|x_n|\} = |\alpha|\, ||\mathbf{x}||_\infty. \end{aligned}$$

N3: Let $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. Then by the usual triangle inequality we have

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||_1 &= \sum_{k=1}^{n} |x_k + y_k| \leqslant \sum_{k=1}^{n} |x_k| + \sum_{k=1}^{n} |y_k| = ||\mathbf{x}||_1 + ||\mathbf{y}||_1; \\ ||\mathbf{x}+\mathbf{y}||_\infty &= \max\{|x_1+y_1|,\ldots,|x_n+y_n|\} \leqslant \max\{|x_1|,\ldots,|x_n|\} + \max\{|y_1|,\ldots,|y_n|\} = ||\mathbf{x}||_\infty + ||\mathbf{y}||_\infty. \end{aligned}$$

Recall that the parallelogram law states that for any $\mathbf{v}$,$\mathbf{w}$ we have

$$||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2 = 2\,||\mathbf{v}||^2 + 2\,||\mathbf{w}||^2.$$

If we choose $\mathbf{v} = (1,0)$, $\mathbf{w} = (0,1)$ in $\mathbb{R}^2$ then

$$\begin{aligned} ||\mathbf{v}+\mathbf{w}||_1^2 + ||\mathbf{v}-\mathbf{w}||_1^2 &= 2^2 + 2^2 = 8; \\ 2\,||\mathbf{v}||_1^2 + 2\,||\mathbf{w}||_1^2 &= 2 \times 1^2 + 2 \times 1^2 = 4, \end{aligned}$$

whilst

$$\begin{aligned} ||\mathbf{v}+\mathbf{w}||_\infty^2 + ||\mathbf{v}-\mathbf{w}||_\infty^2 &= 1^2 + 1^2 = 2; \\ 2\,||\mathbf{v}||_\infty^2 + 2\,||\mathbf{w}||_\infty^2 &= 2 \times 1^2 + 2 \times 1^2 = 4. \end{aligned}$$

Hence neither of these norms satisfy the parallelogram law.