Solution (#971) For $m \times n$ matrices we can define

$$A \cdot B = \text{trace}(B^T A).$$

Let P, Q, R be $m \times n$ matrices and α, β be real numbers.
IP1: Note that

$$\begin{aligned}(\alpha P + \beta Q) \cdot R &= \text{trace}\left(R^T (\alpha P + \beta Q)\right) \\ &= \alpha\text{trace}\left(R^T P\right) + \beta\text{trace}\left(R^T Q\right) \\ &= \alpha (P \cdot R) + \beta (Q \cdot R).\end{aligned}$$

IP2: Now for a square matrix M we have $\text{trace}(M^T) = \text{trace}(M)$. As $B^T A$ is $n \times n$ we have

$$\begin{aligned}P \cdot Q &= \text{trace}(Q^T P) \\ &= \text{trace}(\left(Q^T P\right)^T) \\ &= \text{trace}(P^T Q^{TT}) \\ &= \text{trace}(P^T Q) \\ &= Q \cdot P.\end{aligned}$$

IP3: Finally for an $m \times n$ matrix P we have

$$\begin{aligned}P \cdot P &= \text{trace}(P^T P) \\ &= \sum_{i=1}^{n} \left[P^T P\right]_{ii} \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{m} \left[P^T\right]_{ik} [P]_{ki} \right) \\ &= \sum_{i=1}^{n} \sum_{k=1}^{m} \left([P]_{ki}\right)^2 \geqslant 0\end{aligned}$$

and we have $P \cdot P = 0$ if and only if $[P_{ki}] = 0$ for each k, i. That is, if and only if $P = 0$.

Note generally that

$$P \cdot Q = \text{trace}(Q^T P) = \sum_{i=1}^{n} \left(\sum_{k=1}^{m} [P_{ki}] [Q]_{ki} \right).$$

This is the same as the dot product on $\mathbb{R}^{n^2}$ when we identify a $n \times n$ matrix P with

$$\left([P]_{11}, \ldots, [P]_{1n}, [P]_{21}, , \ldots, [P]_{2n}, \ldots, [P]_{m1}, , \ldots, [P]_{mn}\right).$$