Solution (#972) (i) Let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
where $a, b, c, d > 0$. Then $c_A(x) = x^2 - (a+d)x + (ad - bc)$. As this quadratic has discriminant
$$(a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc > 0$$
then A has two distinct real eigenvalues r, s where $s < r$. Further as $r + s = a + d > 0$ then r is positive.

(ii) Also as $r + s > 0$ even if $s < 0$ it is the case that $|s| = -s < r$.

(iii) For our matrix, and using results from #969 we have
$$\|A\|_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}| = \max \{a + b, c + d\}.$$
If $a + b \geqslant c + d$ then we have
$$r = \frac{(a+d) + \sqrt{(a-d)^2 + 4bc}}{2} \leqslant a + b \qquad \iff \qquad c + d \leqslant b + a.$$
And a similar argument follows if $c + d \geqslant a + b$ that $r \leqslant c + d$.

(iv) r is not a repeated root as we have already shown that the roots are distinct.

(v) With r as above, the r-eigenspace is the null space of $rI_2 - A$ which is spanned by $\mathbf{v} = (b, r-a)^T$. As
$$r > \frac{(a+d) + (a-d)}{2} = a$$
then this r-eigenvector has all-positive co-ordinates.

(vi) Say that $\mathbf{x}$ is an s-eigenvector which is a multiple of $(b, s-a)^T$. As $s - a < 0$ then any s-eigenvector has co-ordinates with different signs.

(vii) For
$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$
and noting $(r-a)(s-a) = -bc$, we similarly have $\mathbf{w} = (c, r-a)^T$. We then have
$$P = \frac{\mathbf{v}\mathbf{w}^T}{\mathbf{w}^T\mathbf{v}} = \frac{1}{(r-a)^2 + bc} \begin{pmatrix} bc & b(r-a) \\ c(r-a) & (r-a)^2 \end{pmatrix}.$$
Note further that as the quadratic $c_A(x)$ has r and s as roots, we have $rs = ad - bc$ and $r + s = a + d$ and hence
$$(r-a)(s-a) = -bc \qquad \text{and} \qquad (r-a)(r-s) = (r-a)^2 + bc.$$
So the above matrix simplifies to
$$P = \frac{1}{(r-a)(r-s)} \begin{pmatrix} (r-a)(a-s) & b(r-a) \\ c(r-a) & (r-a)^2 \end{pmatrix} = \frac{1}{r-s} \begin{pmatrix} a-s & b \\ c & r-a \end{pmatrix}.$$
Now with
$$X = \begin{pmatrix} b & b \\ r-a & s-a \end{pmatrix} \qquad \text{so that} \qquad X^{-1} = \frac{1}{b(s-r)} \begin{pmatrix} s-a & -b \\ a-r & b \end{pmatrix},$$
and $P^{-1}AP = \mathrm{diag}(r, s)$ we have
$$\frac{A^k}{r^k} = \frac{1}{r^k} \left(X \mathrm{diag}(r^k, s^k) X^{-1} \right) = X \mathrm{diag}\left(1, \frac{s^k}{r^k}\right) X^{-1}.$$
As $|s| < r$ then in the limit A^k/r^k approximates as required to
$$X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1} = \frac{1}{b(s-r)} \begin{pmatrix} b(s-a) & -b^2 \\ (r-a)(s-a) & b(a-r) \end{pmatrix} = \frac{1}{r-s} \begin{pmatrix} a-s & b \\ c & r-a \end{pmatrix}.$$
(viii) The second column of P is clearly a multiple of $\mathbf{v}$ and we also note
$$\begin{pmatrix} a-s \\ c \end{pmatrix} = \frac{c}{r-a} \begin{pmatrix} b \\ r-a \end{pmatrix}$$
as $(r-a)(s-a) = -bc$. So P has rank one and column space spanned by $\mathbf{v}$. Finally P is a projection matrix as
$$P^2 = \left(\frac{\mathbf{v}\mathbf{w}^T}{\mathbf{w}^T\mathbf{v}}\right)\left(\frac{\mathbf{v}\mathbf{w}^T}{\mathbf{w}^T\mathbf{v}}\right) = \frac{\mathbf{v}(\mathbf{w}^T\mathbf{v})\mathbf{w}^T}{(\mathbf{w}^T\mathbf{v})^2} = \frac{\mathbf{v}\mathbf{w}^T}{\mathbf{w}^T\mathbf{v}} = P.$$