Solution (#973) Let V be the set of polynomials $p(x)$ defined on the interval $a \leqslant x \leqslant b$. We define

$$p \cdot q = \int_a^b p(x)q(x)\,\mathrm{d}x.$$

Let α, β be real scalars and p, q, r be polynomials.
IP1: Note that

$$\begin{aligned}
(\alpha p + \beta q) \cdot r &= \int_a^b (\alpha p(x) + \beta q(x))r(x)\,\mathrm{d}x \\
&= \alpha \int_a^b p(x)r(x)\,\mathrm{d}x + \beta \int_a^b q(x)r(x)\,\mathrm{d}x \\
&= \alpha(p \cdot r) + \beta\,(q \cdot r)\,.
\end{aligned}$$

IP2: We also have

$$p \cdot q = \int_a^b p(x)q(x)\,\mathrm{d}x = \int_a^b q(x)qp(x)\,\mathrm{d}x = q \cdot p.$$

IP3: And

$$p \cdot p = \int_a^b p(x)^2\,\mathrm{d}x \geqslant 0$$

as the integrand is non-negative. In fact if $p(x_0) \neq 0$ at some particular point $a \leqslant x_0 \leqslant b$ then there will be an interval I of non-zero length containing x_0 on which $p(x) > 0$ or on which $p(x) < 0$. Then

$$\int_a^b p(x)^2\,\mathrm{d}x \geqslant \int_I p(x)^2\,\mathrm{d}x > 0.$$

Hence $p \cdot p = 0$ implies $p = 0$.

Remark: This final argument essentially relies on the continuity of polynomials. An alternative proof could be made using Legendre polynomials and #974. Let $a = -1$ and $b = -1$, and note that a polynomial p of degree n can be written as

$$p(x) = a_n P_n(x) + a_{n-1}P_{n-1}(x) + \cdots + a_0 P_0(x).$$

It follows from#974 that

$$p \cdot p = |a_n|^2\, P_n \cdot P_n + \cdots + |a_0|^2\, P_0 \cdot P_0.$$

If $p \cdot p = 0$ then each $a_k = 0$ and so $p = 0$.

More generally, for other a and b, the polynomials

$$P_n\left(\frac{2x - a - b}{b - a}\right)$$

are similarly orthogonal for this inner product and the same argument can be made.