Solution (#974) Let $P_n(x)$ denote the nth Legendre polynomial defined on the interval $-1 \leqslant x \leqslant 1$. Then $P_n(x)$ is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = -n(n+1)y.$$

Then by IBP twice we have

$$\begin{aligned} -n(n+1)P_n \cdot P_m &= -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x \\ &= \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)P_n'(x)\right)P_m(x)\,\mathrm{d}x \\ &= \left[(1-x^2)P_n'(x)P_m(x)\right]_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,\mathrm{d}x. \end{aligned}$$

Swapping the roles of m and n we then see

$$n(n+1)P_n \cdot P_m = m(m+1)P_n \cdot P_m.$$

If $n \neq m$, say $n > m$, then $n(n+1) > m(m+1)$ and hence $P_n \cdot P_m = 0$.

Now by Rodrigues' Formula we have

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right].$$

So

$$2^{2n}(n!)^2 P_n \cdot P_n = \int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right]\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right]\mathrm{d}x.$$

If we apply IBP we find that the RHS equals

$$\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[\left(x^2-1\right)^n\right]\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right]\mathrm{d}x\right]_{-1}^{1} - \int_{-1}^{1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[\left(x^2-1\right)^n\right]\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[\left(x^2-1\right)^n\right]\mathrm{d}x.$$

However note that as $(x^2-1)^n = (x-1)^n(x+1)^n$ then, by Leibniz's theorem for the derivatives of products,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\left(x^2-1\right)^n\right] = 0 \qquad \text{at } x=1 \text{ and } x=-1$$

for $0 \leqslant k < n$ as at least one factor of $x-1$ and $x+1$ will still remain. Hence after applying IBP n times we arrive at

$$2^{2n}(n!)^2 P_n \cdot P_n = (-1)^n\int_{-1}^{1}\left(x^2-1\right)^n\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left[\left(x^2-1\right)^n\right]\mathrm{d}x.$$

Now $(x^2-1)^n$ is a polynomial of degree $2n$ whose leading coefficient is 1. Hence

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left[\left(x^2-1\right)^n\right] = (2n)!.$$

So we now have

$$P_n \cdot P_n = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}\left(x^2-1\right)^n\,\mathrm{d}x.$$

Say now we set $x = \sin t$ as a substitution in the integral. Then

$$\begin{aligned} P_n \cdot P_n &= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-\pi/2}^{\pi/2}\left(\sin^2 t - 1\right)^n\cos t\,\mathrm{d}t \\ &= \frac{(2n)!}{2^{2n-1}(n!)^2}\int_{0}^{\pi/2}\cos^{2n+1} t\,\mathrm{d}t \\ &= \frac{(2n)!}{2^{2n-1}(n!)^2} \times \frac{2^{2n}(n!)^2}{(2n+1)!} \qquad \text{[by \#263]} \\ &= \frac{2}{(2n+1)}. \end{aligned}$$