Solution (#979) Let

$$X = \langle (4,2,3,3), (1,0,1,1), (1,2,0,0) \rangle \qquad \text{and} \qquad Y = \{(y_1, y_2, y_3, y_4) : y_1 + y_2 = y_3 + y_4 = 0\},$$

be subspaces of $\mathbb{R}^4$.

We may find the dimension of X by row-reducing the three given vectors that span X. We see

$$\begin{pmatrix} 4 & 2 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 2 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

to see that $\dim X = 2$. In particular a general vector of X has the form $(\alpha, 2\beta, \alpha - \beta, \alpha - \beta)$.

Now a general vector of Y is of the form $(\gamma, -\gamma, \delta, -\delta)$ showing that $\dim Y = 2$. The vector $(\gamma, -\gamma, \delta, -\delta)$ lies in X as well if

$$\gamma = \alpha, \quad -\gamma = 2\beta, \quad \delta = \alpha - \beta, \quad -\delta = \alpha - \beta.$$

So $\delta = 0$ and $\alpha = \beta$ from the last two equations. But then $\gamma = \alpha = \beta$ from the first two equations. Hence $X \cap Y = \{0\}$.

Finally as $(1,0,1,1), (0,2,-1,-1)$ span X and $(1,-1,0,0), (0,0,1,-1)$ span Y then $X + Y$ is spanned by these four vectors. Row-reducing them gives

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to I_4.$$

Hence we have

$$\dim X = 2 = \dim Y, \qquad \dim(X \cap Y) = 0, \qquad \dim(X + Y) = 4,$$

and (3.65) holds as $4 = 2 + 2 - 0$.