Solution (#981) (i) $V = \mathbb{R}^3$, $X_1 = \{(x,x,z) : x, z \in \mathbb{R}\}$, $X_2 = \{(x,y,y) : x, y \in \mathbb{R}\}$. Note that the vector $(1,1,1)$ belongs to both X_1 and X_2 and hence $X_1 \cap X_2 \neq 0$. In particular by #984 V is not a direct sum of X_1 and X_2.

(ii) $V = \mathbb{R}^3$, $X_1 = \{(x,0,x) : x \in \mathbb{R}\}$, $X_2 = \{(x,y,y) : x, y \in \mathbb{R}\}$. Note that for any (x,y,z) in $\mathbb{R}^3$ we have

$$(x,y,z) = (z-y, 0, z-y) + (x+y-z, y, y)$$

and so $\mathbb{R}^3 = X_1 + X_2$, Further if (x,y,z) lies in both X_1 and X_2 then

$$x = z, \qquad y = 0, \qquad y = z$$

and so $(x,y,z) = \mathbf{0}$. Hence $\mathbb{R}^3 = X_1 \oplus X_2$.

(iii) $V = \mathbb{R}^3, X_1 = \{(x,0,x) : x \in \mathbb{R}\}$, $X_2 = \{(0,x,x) : x \in \mathbb{R}\}$. Now note that the vector $(1,0,0)$ lies outside $X_1 + X_2$ for if

$$(1,0,0) = (x,0,x) + (0,y,y)$$

then

$$1 = x, \qquad 0 = y, \qquad 0 = x + y$$

which are contradictory.

(iv) $V = \mathbb{R}^3, X_1 = \{(x,0,x) : x \in \mathbb{R}\}$, $X_2 = \{(0,x,x) : x \in \mathbb{R}\}$, $X_3 = \{(x,x,x) : x \in \mathbb{R}\}$. Say now that (x,y,z) can be written

$$(x,y,z) = (a,0,a) + (0,b,b) + (c,c,c).$$

Then

$$x = a + c, \qquad y = b + c, \qquad z = a + b + c$$

and so it must necessarily follow that

$$a = z - y, \qquad b = z - x, \qquad c = x + y - z$$

and it is an easy check that these x, y, z are indeed solutions. Hence $\mathbb{R}^3 = X_1 \oplus X_2 \oplus X_3$.