Solution (#984) Let V, X, Y be subspaces of $\mathbb{R}^n$. Recall that we write $V = X \oplus Y$ if and only if every $\mathbf{v}$ in V can be uniquely written as $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x}$ is in X and $\mathbf{y}$ is in Y,

Say that $V = X \oplus Y$. Then certainly $V = X + Y$. Further if $\mathbf{v}$ is in $X \cap Y$ then

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v}$$

are two expressions for $\mathbf{v}$ as a sum of elements of X and Y. By uniqueness it must be the case that $\mathbf{v} = \mathbf{0}$ as required.

Conversely say that $V = X + Y$ and $X \cap Y = \{\mathbf{0}\}$. Then every $\mathbf{v}$ in V can be written as $\mathbf{v} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{x}$ is in X and $\mathbf{y}$ is in Y, If

$$\mathbf{v} = \mathbf{x}_1 + \mathbf{y}_1 = \mathbf{x}_2 + \mathbf{y}_2$$

are two such expressions, then

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{y}_2 - \mathbf{y}_1.$$

Note that the LHS is an element of X and RHS is an element of Y. Thus the above element is in $X \cap Y$ and so equals $\mathbf{0}$. Therefore $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{y}_1 = \mathbf{y}_2$, showing that every $\mathbf{v}$ in V can be uniquely written $\mathbf{v} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{x}$ is in X and $\mathbf{y}$ is in Y,