Solution (#987) Let V be a subspace of $\mathbb{R}^n$ such that

$$V = X_1 \oplus X_2 \oplus \cdots \oplus X_k.$$

This means that every $\mathbf{v}$ in V can be uniquely written as

$$\mathbf{v} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k \qquad \text{where } \mathbf{x}_i \text{ is in } X_i.$$

Say now that $\mathcal{B}_i$ is a basis for X_i, for each i, and define

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k.$$

For any $\mathbf{v}$ in V we may write

$$\mathbf{v} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k \qquad \text{where } \mathbf{x}_i \text{ is in } X_i.$$

As $\mathcal{B}_i$ is a basis for X_i then each $\mathbf{x}_i$ can be expressed as a linear combination of the elements of $\mathcal{B}_i$. By the above it follows that $\mathbf{v}$ is a linear combination of the elements of $\mathcal{B}$. That is, $\mathcal{B}$ spans V.

Say now that

$$\sum_j \alpha_j \mathbf{b}_j = \mathbf{0}$$

where the $\mathbf{b}_j$ are elements of $\mathcal{B}$. Each $\mathbf{b}_j$ is an element of some $\mathcal{B}_i$ and so the above sum can be separated into sums

$$\sum_{i=1}^k \sum_i \alpha_j^i \mathbf{b}_j^i = \mathbf{0}.$$

The individual sum

$$\sum_i \alpha_j^i \mathbf{b}_j^i$$

lies in X_i and as

$$\mathbf{0} = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0}$$

is the only way to express $\mathbf{0}$ as a sum of elements in each X_i, it follows that

$$\sum_i \alpha_j^i \mathbf{b}_j^i = \mathbf{0}$$

for each i. Finally as $\mathcal{B}_i$ is a basis for X_i then each $\alpha_j^i = 0$ by independence. Hence $\mathcal{B}$ is also independent.

Finally note that $|\mathcal{B}_i| = \dim X_i$ for each i and that $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint for each $i \neq j$. (Any vector $\mathbf{v}$ belonging to both could be written in two different ways as a sum of elements in $X_1, X_2, \ldots, X_k$.) Hence we have

$$\dim V = |\mathcal{B}| = \sum_{i=1}^k |\mathcal{B}_i| = \sum_{i=1}^k \dim X_i$$

as required.