Solution (#992) Let A be an $n \times n$ matrix. Say that the distinct eigenvalues of A are $\lambda_1, \ldots, \lambda_k$ with corresponding eigenspaces $E_1, \ldots, E_k$. (Note we are not assuming that $c_A(x)$ has no repeated roots, just that the above is a list of those roots not including any repetitions.) Say that

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k = \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k$$

where $\mathbf{v}_i$ and $\mathbf{w}_i$ are in the ith eigenspace. Then

$$(\mathbf{v}_1 - \mathbf{w}_1) + (\mathbf{v}_2 - \mathbf{w}_2) + \cdots + (\mathbf{v}_k - \mathbf{w}_k) = \mathbf{0}.$$

Now $\mathbf{v}_i - \mathbf{w}_i$ are in the ith eigenspace and these eigenvectors have distinct eigenvalues. As distinct eigenvalue eigenvectors are linearly independent (Proposition 3.194) then it must be that $\mathbf{v}_i - \mathbf{w}_i = \mathbf{0}$ for each i. In particular any expression of a vector in

$$E_1 + E_2 + \cdots + E_k$$

as a summand $\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k$ is unique and so

$$W = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

is a direct sum.

If W is a proper subspace of $\mathbb{R}_n$ then it will be impossible to find n independent eigenvectors. However if $W = \mathbb{R}_n$ then the union of bases from each E_i will be a basis of $\mathbb{R}_n$ (by #987) and, as each vector is an eigenvector then we have created an eigenbasis and A is diagonalizable.