Solution (#998) (i) Let U be an upper triangular $n \times n$ matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_n$. Note that $xI - U$ is upper triangular and hence $c_U(x) = \det(xI - U)$ is the product of the diagonal entries of $xI - U$. Hence

$$c_U(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n).$$

(ii) Note that the first column of U is $\alpha_1 \mathbf{e}_1^T$ and so the first column of $U - \alpha_1 I$ is zero. Hence we have

$$(U - \alpha_1 I)\mathbf{e}_1^T = \mathbf{0}.$$

Considering the second column of U, note

$$U\mathbf{e}_2^T = [U]_{12}\mathbf{e}_1^T + \alpha_2 \mathbf{e}_2^T.$$

So

$$(U - \alpha_2 I)\mathbf{e}_2^T = [U]_{12}\mathbf{e}_1^T$$

and so

$$(U - \alpha_1 I)(U - \alpha_2 I)\mathbf{e}_2^T = (U - \alpha_1 I)\left([U]_{12}\mathbf{e}_1^T\right) = [U]_{12}(U - \alpha_1 I)\mathbf{e}_1^T = \mathbf{0}.$$

Continuing in this vein we can see that

$$(U - \alpha_i I)\mathbf{e}_i^T \quad \text{is in the span} \quad \langle \mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_{i-1}^T \rangle;$$
$$(U - \alpha_{i-1} I)(U - \alpha_i I)\mathbf{e}_i^T \quad \text{is in the span} \quad \langle \mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_{i-2}^T \rangle;$$

and so on to

$$(U - \alpha_2 I) \cdots (U - \alpha_i I)\mathbf{e}_i^T \quad \text{is in the span} \quad \langle \mathbf{e}_1^T \rangle;$$
$$(U - \alpha_1 I)(U - \alpha_2 I) \cdots (U - \alpha_i I)\mathbf{e}_i^T = \mathbf{0}.$$

(iii) Let $1 \leqslant j \leqslant i$. As polynomials in U commute then

$$(U - \alpha_1 I)(U - \alpha_2 I) \cdots (U - \alpha_i I)\mathbf{e}_j^T = \mathbf{0}$$

as

$$(U - \alpha_1 I)(U - \alpha_2 I) \cdots (U - \alpha_j I)\mathbf{e}_j^T = \mathbf{0}.$$

Hence taking $i = n$ we have that

$$c_U(U)\mathbf{e}_j^T = (U - \alpha_1 I)(U - \alpha_2 I) \cdots (U - \alpha_n I)\mathbf{e}_j^T = \mathbf{0}$$

for each $j = 1, \ldots, n$. This being true on a basis we then have $c_U(U) = 0$ by linearity. That is the Cayley-Hamilton theorem for upper triangular matrices.

(iv) Now let A be an $n \times n$ matrix (real or complex). By #996 we know that there is a complex $n \times n$ matrix P such that $P^{-1}AP = U$ is upper triangular. By #919 we know that $c_A(x) = c_U(x)$. By #618 and (iii) above we then have

$$c_A(A) = c_A(PUP^{-1}) = Pc_A(U)P^{-1} = Pc_U(U)P^{-1} = P(0)P^{-1},$$

and so the Cayley-Hamilton theorem follows for all $n \times n$ matrices.