Solution (#1005) Say that A is an upper triangular matrix and say that the diagonal entries of A are $\alpha_1, \alpha_2, \ldots, \alpha_n$. Note that for any $1 \leqslant r \leqslant n$

$$(A - \alpha_r I)\mathbf{e}_r^T \quad \text{is in} \quad \langle \mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_{r-1}^T \rangle.$$

So for any $1 \leqslant s \leqslant r \leqslant n$

$$(A - \alpha_1 I)(A - \alpha_2 I) \cdots (A - \alpha_r I)\mathbf{e}_s^T = \mathbf{0}$$

as shown in the solution of #998. It follows that

$$(A - \alpha_1 I)(A - \alpha_2 I) \cdots (A - \alpha_n I)\mathbf{e}_s^T = \mathbf{0} \qquad \text{for } 1 \leqslant s \leqslant n.$$

By #1001 we know that $m_A(x)$ divides $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$. As the latter polynomial is a product of linear factors then so is any polynomial that divides it. Hence $m_A(x)$ is a product of linear factors.

As similar matrices have equal minimal polynomials (#706) then a triangularizable matrix has a minimal polynomial which is a product of linear factors.