Solution (#1007) Say that a matrix A has a minimal polynomial $m(x)$ which is a product of linear factors (not necessarily distinct). If

$$m(x) = (x - a_1)^{r_1}(x - a_2)^{r_2} \cdots (x - a_k)^{r_k}$$

where $a_1, \ldots, a_k$ are distinct real numbers and $r_1, r_2, \ldots, r_k$ are positive integers. Now $\mathrm{Null}(m(A)) = \mathbb{R}_n$ and by #997 we have

$$\mathbb{R}_n = \mathrm{Null}((A - a_1 I)^{r_1}) \oplus \mathrm{Null}((A - a_2 I)^{r_2}) \oplus \cdots \oplus \mathrm{Null}(A - a_k I)^{r_k}).$$

For ease we will consider the case $k = 3$ so that we can address the full details of the proof without too much issue notationally.. Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be a basis for $\mathrm{Null}((A - a_1 I)^{r_1})$, $\mathbf{y}_1, \ldots, \mathbf{y}_s$ be a basis for $\mathrm{Null}((A - a_2 I)^{r_2})$ and $\mathbf{z}_1, \ldots, \mathbf{z}_t$ be a basis for $\mathrm{Null}((A - a_3 I)^{r_3})$. Their union is then a basis for $\mathbb{R}_n$. If we let

$$P = (\mathbf{x}_1 \,|\, \cdots \,|\, \mathbf{x}_r \,|\, \mathbf{y}_1 \,|\, \cdots \,|\, \mathbf{y}_s \,|\, \mathbf{z}_1 \,|\, \cdots \,|\, \mathbf{z}_t)$$

then

$$P^{-1}AP = \mathrm{diag}(A_1, A_2, A_3).$$

We have in each case that

$$(A_i - a_i I)^{r_i} = 0.$$

So by #746 there are invertible matrices Q_1, Q_2, Q_3 such that

$$Q_i^{-1}(A_i - a_i I)Q_i = Q_i^{-1}A_i Q_i - a_i I$$

is strictly upper triangular. Hence each $Q_i^{-1}A_i Q_i$ is upper triangular. If we set $Q = \mathrm{diag}(Q_1, Q_2, Q_3)$ then we find that

$$(PQ)^{-1}A(PQ) = Q^{-1}P^{-1}APQ = \mathrm{diag}(Q_1^{-1}A_1Q_1, Q_2^{-1}A_2Q_2, Q_3^{-1}A_3Q_3)$$

is upper triangular.