Solution (#1008) Say that A and B are diagonalizable. Then their minimal polynomials m_A and m_B are products of distinct linear factors. Say

$$\begin{aligned} m_A(x) &= (x-\alpha_1)\cdots(x-\alpha_r)(x-\beta_1)\cdots(x-\beta_s); \\ m_B(x) &= (x-\alpha_1)\cdots(x-\alpha_r)(x-\gamma_1)\cdots(x-\gamma_s), \end{aligned}$$

where $\alpha_1,\ldots,\alpha_r$ are the common roots of m_A and m_B and so

$$\alpha_1,\ldots\alpha_r,\beta_1,\ldots\beta_s,\gamma_1,\ldots\gamma_t$$

are distinct. Then I claim $C = \mathrm{diag}(A,B)$ has minimal polynomial

$$m_C(x) = (x-\alpha_1)\cdots(x-\alpha_r)(x-\beta_1)\cdots(x-\beta_s)(x-\gamma_1)\cdots(x-\gamma_s).$$

Certainly

$$m_C(A) = 0 = m_C(B)$$

as m_A and m_B both divide m_C. However if we are to have

$$p(C) = \mathrm{diag}(p(A), p(B)) = 0$$

it must be the case that m_A and m_B both divide $p(x)$. As $p(x)$ is the least common multiple of m_A and m_B then no proper factor of m_C has C as a root.