Solution (#1010) (i) Say that A and B are simultaneously diagonalizable, Then there exists a matrix P such that $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$ are both diagonal. As diagonal matrices commute then

$$\left(P^{-1}AP\right)\left(P^{-1}BP\right) = (P^{-1}BP)(P^{-1}AP).$$

This simplifies to $AB = BA$.

(ii) Say now that A and B commute. If $\mathbf{v}$ is a λ-eigenvector of A then

$$A\left(B\mathbf{v}\right) = BA\mathbf{v} = B\lambda\mathbf{v} = \lambda\left(B\mathbf{v}\right)$$

and so $B\mathbf{v}$ is also a λ-eigenvector.

(iii) Say now that A and B are commuting diagonalizable matrices. As A is diagonalizable then we have

$$\mathbb{R}_n = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

where $E_1, \ldots, E_k$ are the eigenspaces of A. If we take a basis for each eigenspace, then their union is a basis for $\mathbb{R}_n$ by #992. Further if we make those vectors the columns of a matrix P we have

$$P^{-1}AP = \operatorname{diag}(\lambda_1 I, \lambda_2 I, \ldots, \lambda_k I)$$

and as each E_i is B-invariant we also have

$$P^{-1}BP = \operatorname{diag}(B_1, B_2, \ldots, B_k)$$

for certain matrices $B_1, \ldots, B_k$. Now B is diagonalizable and so by #706 and #1004 $m_B(x) = m_{P^{-1}BP}(x)$ is a product of distinct linear factors. As

$$p(\operatorname{diag}(B_1, B_2, \ldots, B_k)) = \operatorname{diag}(p(B_1), p(B_2), \ldots, p(B_k))$$

for any polynomial $p(x)$ then it follows that

$$m_B(B_i) = 0 \qquad \text{for each } i$$

and in particular $m_{B_i}(x)$ divides $m_B(x)$ and so too is a product of distinct linear factors; so again by #1004 B_i is diagonalizable. So there are invertible matrices Q_i such that $Q_i^{-1}B_iQ_i$ is diagonal and if we set

$$Q = \operatorname{diag}(Q_1, Q_2, \ldots, Q_k)$$

then

$$\begin{aligned}(PQ)^{-1}A(PQ) &= \operatorname{diag}(Q_1^{-1}(\lambda_1 I)Q_1, \ldots, Q_k^{-1}(\lambda_k I)Q_k) = \operatorname{diag}(\lambda_1 I, \ldots, \lambda_k I)\\ (PQ)^{-1}B(PQ) &= \operatorname{diag}(Q_1^{-1}B_1Q_1, \ldots, Q_k^{-1}B_kQ_k)\end{aligned}$$

are both diagonal and A and B are simultaneously diagonalizable.