Solution (#1012) The adjacency matrices A_i for each of the graphs G_i are

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then $c_{A_1}(x) = (x+1)^2(x-2)$ and $c_{A_2}(x) = x(x^2-2)$.

Note that A_1 has an eigenbasis

$$(1,-1,0)^T, \qquad (0,1,-1)^T, \qquad (1,1,1)^T$$

and A_2 has an eigenbasis

$$(0,1,-1)^T, \qquad \left(\sqrt{2},-1,-1\right)^T, \qquad \left(\sqrt{2},1,1\right)^T.$$

Hence

$$\begin{aligned} A_1^n &= \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{3} \begin{pmatrix} 2(-1)^n + 2^n & (-1)^{n+1} + 2^n & (-1)^{n+1} + 2^n \\ (-1)^{n+1} + 2^n & 2(-1)^n + 2^n & (-1)^{n+1} + 2^n \\ (-1)^{n+1} + 2^n & (-1)^{n+1} + 2^n & 2(-1)^n + 2^n \end{pmatrix}. \end{aligned}$$

Similarly

$$\begin{aligned} A_2^n &= \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^n 2^{n/2} & 0 \\ 0 & 0 & 2^{n/2} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} (1+(-1)^n)\,2^{(n+2)/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} \\ \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & 2+(1+(-1)^n)\,2^{n/2} & -2+(1+(-1)^n)\,2^{n/2} \\ \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & -2+(1+(-1)^n)\,2^{n/2} & 2+(1+(-1)^n)\,2^{n/2} \end{pmatrix}. \end{aligned}$$

We can treat A_3 in the same way or note that

$$A_3 = P^{-1} A_2 P \qquad \text{where} \qquad P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

as G_2 and G_3 are isomorphic graphs. Here P is the permutation matrix interchanging v_1 and v_3. Then

$$\begin{aligned} A_3^n &= P^{-1} A_2^n P \\ &= \frac{1}{4} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} (1+(-1)^n)\,2^{(n+2)/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} \\ \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & 2+(1+(-1)^n)\,2^{n/2} & -2+(1+(-1)^n)\,2^{n/2} \\ \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & -2+(1+(-1)^n)\,2^{n/2} & 2+(1+(-1)^n)\,2^{n/2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2+(1+(-1)^n)\,2^{n/2} & -2+(1+(-1)^n)\,2^{n/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} \\ -2+(1+(-1)^n)\,2^{n/2} & 2+(1+(-1)^n)\,2^{n/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} \\ \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & \left(1+(-1)^{n+1}\right)2^{(n+1)/2} & (1+(-1)^n)\,2^{(n+2)/2} \end{pmatrix}. \end{aligned}$$