Solution (#1015) (i) If there is an Euler circuit then it enters and leaves every vertex v a number of times, k say, and so $\deg(v) = 2k$ is even. To prove the converse we will use induction on the number of edges E. Take a graph G with each vertex having even degree and E edges and assume that each such graph with fewer edges has an Euler circuit. As every vertex is connected by some edge then $\deg(v) \geqslant 2$ for all vertices and so G contains a cycle C by #1014. The graph $G - C$ obtained from G, by removing the edges of C, still has vertices of even degree. It is possibly no longer connected, but we can apply our induction to each of its component parts separately as they have fewer than E edges. Thus we can create an Euler circuit for G by going around the cycle C, interspersing our journey with these Euler circuits as we pass through each component.

(ii) Assume the graph is traversable. If the path traversing the graph is a cycle then every vertex has even degree by (i) and no vertex has odd degree. If there is a path, which is not a cycle, which traverses the graph then the path starts and ends in two distinct points. Every vertex is somewhere on the path, and every vertex other than the start and end is arrived at and departed from multiples times. The degree of such a vertex is $2n$, and so even, where n is the number of times the path visits that vertex. In a similar manner the degree of the vertex at the start of the path is $1 + 2m$, and so odd, where m is the number of times that the path is revisited during the path. The vertex at the end of the path similarly has an odd degree.

Say now that a graph has no or two vertices of odd degree. If it has none then the graph has an Euler circuit by (i) and so the graph is traversable. If the graph has two odd degree vertices then we can create a new graph by adding an edge between those two odd degree vertices. The vertices of this new graph are all of even degree and so by (i) there is an Euler circuit for the new graph. If we omit the added edge from this Euler circuit, then we have a traversing path of the original graph.