Solution (#1027) Let A be the adjacency matrix of a bipartite graph with vertices $v_1, \ldots, v_n$ As the graph is bipartite we can partition the vertex set into disjoint subsets V_1 and V_2 such that every edge of the graph connects a vertex in V_1 to one in V_2. Say that $v_{i_1}, \ldots, v_{i_m}$ are the vertices in V_1 and let P be any permutation matrices whose first m columns are

$$\mathbf{e}_{i_1}^T, \mathbf{e}_{i_2}^T, \ldots, \mathbf{e}_{i_m}^T.$$

Denote the remaining columns of P as $\mathbf{e}_{i_{m+1}}^T, \mathbf{e}_{i_{m+2}}^T, \ldots, \mathbf{e}_{i_n}^T$.

We have for any $1 \leqslant k \leqslant m$ that

$$AP\mathbf{e}_k^T = A\mathbf{e}_{i_k}^T = \sum_{r=m+1}^{n} b_{rk}\mathbf{e}_{i_r}^T = \sum_{r=m+1}^{n} b_{rk}P\mathbf{e}_r^T$$

for some b_{ri} as the graph is bipartite. So

$$P^{-1}AP\mathbf{e}_k^T = \sum_{r=m+1}^{n} b_{rk}\mathbf{e}_r^T.$$

In a similar fashion we have for $m+1 \leqslant k \leqslant n$ that

$$P^{-1}AP\mathbf{e}_k^T = \sum_{r=1}^{m} b_{rk}\mathbf{e}_r^T.$$

Finally as P is orthogonal and A is symmetric then $P^{-1}AP = P^TAP$ is symmetric and we have

$$P^{-1}AP = \begin{pmatrix} 0_{mm} & B \\ B^T & 0_{nn} \end{pmatrix}$$

for some $m \times (n-m)$ matrix B.

The eigenvalues of A are the same as those of $P^{-1}AP$. If $\lambda = 0$ then there is nothing to prove. If λ is a non-zero eigenvalue of $P^{-1}AP$ this means that there is a vector

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$$

(where $\mathbf{v}_1$ is in $\mathbb{R}_m$ and $\mathbf{v}_2$ is in $\mathbb{R}_{n-m}$) such that

$$P^{-1}AP\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0_{mm} & B \\ B^T & 0_{nn} \end{pmatrix}\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} B\mathbf{v}_2 \\ B^T\mathbf{v}_1 \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{v}_1 \\ \lambda\mathbf{v}_2 \end{pmatrix}.$$

So

$$B\mathbf{v}_2 = \lambda\mathbf{v}_1 \qquad \text{and} \qquad B^T\mathbf{v}_1 = \lambda\mathbf{v}_2.$$

Note also that as $\mathbf{v} \neq \mathbf{0}$ and $\lambda \neq 0$ then $\mathbf{v}_1 \neq \mathbf{0} \neq \mathbf{v}_2$. We then have

$$P^{-1}AP\begin{pmatrix} \mathbf{v}_1 \\ -\mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0_{mm} & B \\ B^T & 0_{nn} \end{pmatrix}\begin{pmatrix} \mathbf{v}_1 \\ -\mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} -B\mathbf{v}_2 \\ B^T\mathbf{v}_1 \end{pmatrix} = \begin{pmatrix} -\lambda\mathbf{v}_1 \\ \lambda\mathbf{v}_2 \end{pmatrix} = -\lambda\begin{pmatrix} \mathbf{v}_1 \\ -\mathbf{v}_2 \end{pmatrix}$$

and so $-\lambda$ is also an eigenvalue.