Solution (#1033) (i) Let x, y and z denote the long-term probabilities of being in states X, Y and Z respectively. Now

$$\begin{aligned} x = P(\text{being in state } X) &= P(\text{previously in state } X \text{ and moving to } X) \\ &\quad +P(\text{previously in state } Y \text{ and moving to } X) \\ &\quad +P(\text{previously in state } Z \text{ and moving to } X) \\ &= \frac{1}{2}x + \frac{1}{3}y + \frac{1}{3}z. \end{aligned}$$

Similarly

$$y = \frac{1}{2}x + \frac{1}{3}z; \qquad z = \frac{2}{3}y + \frac{1}{3}z.$$

Also $x + y + z = 1$ and solving these we find

$$x = \frac{4}{10}, \qquad y = \frac{3}{10}, \qquad z = \frac{3}{10}.$$

(ii) The transition matrix M for the Markov chain equals

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Note that

$$(x, y, z)M = \frac{(4, 3, 3)}{10} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \frac{(4, 3, 3)}{10} = (x, y, z).$$

(iii) The characteristic polynomial of M equals

$$\begin{aligned} c_M(\lambda) &= \begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & \lambda & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \lambda - \frac{1}{3} \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} & 0 \\ \lambda - 1 & \lambda & -\frac{2}{3} \\ \lambda - 1 & -\frac{1}{3} & \lambda - \frac{1}{3} \end{vmatrix} \\ &= (\lambda - 1) \begin{vmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & \lambda & -\frac{2}{3} \\ 1 & -\frac{1}{3} & \lambda - \frac{1}{3} \end{vmatrix} = (\lambda - 1) \begin{vmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \lambda + \frac{1}{2} & -\frac{2}{3} \\ 0 & \frac{1}{6} & \lambda - \frac{1}{3} \end{vmatrix} \\ &= (\lambda - 1) \left[\lambda^2 + \frac{1}{6}\lambda - \frac{1}{18}\right] = (\lambda - 1) \left(\lambda - \frac{1}{6}\right) \left(\lambda + \frac{1}{3}\right). \end{aligned}$$

Three eigenvectors are

$$\text{1-eigenvector } (1, 1, 1)^T; \qquad \frac{1}{6}\text{-eigenvector } (3, -2, -2)^T; \qquad \frac{-1}{3}\text{-eigenvector } (3, -5, 1)^T.$$

So

$$P^{-1}MP = \text{diag}\left(1, \frac{1}{6}, \frac{-1}{3}\right) \qquad \text{where } P = \begin{pmatrix} 1 & 3 & 3 \\ 1 & -2 & -5 \\ 1 & -2 & 1 \end{pmatrix},$$

and

$$\begin{aligned} M^n &= P\text{diag}(1, (1/6)^n, (-1/3)^n)P^{-1} \\ &= \begin{pmatrix} 1 & 3 & 3 \\ 1 & -2 & -5 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/6)^n & 0 \\ 0 & 0 & (-1/3)^n \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 1 & -2 & -5 \\ 1 & -2 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{30} \begin{pmatrix} 12 + 18(1/6)^n & 9 + 6(1/6)^n - 15(-1/3)^n & 9 - 24(1/6)^n + 15(-1/3)^n \\ 12 - 12(1/6)^n & 9 - 4(1/6)^n + 25(-1/3)^n & 9 + 16(1/6)^n - 25(-1/3)^n \\ 12 - 12(1/6)^n & 9 - 4(1/6)^n - 5(-1/3)^n & 9 + 16(1/6)^n + 5(-1/3)^n \end{pmatrix}. \end{aligned}$$

Note as n becomes large that M^n approximates to

$$\begin{pmatrix} 4/10 & 3/10 & 3/10 \\ 4/10 & 3/10 & 3/10 \\ 4/10 & 3/10 & 3/10 \end{pmatrix}$$

showing that long term there is negligible effect resulting from the choice of starting position.