Solution (#1035) (i) Denote the transition matrix as M and say $\mathbf{p} = (p_1, \ldots, p_k)$ is a stationary distribution. Then $\mathbf{p}M = \mathbf{p}$ and so $\mathbf{p}M^n = \mathbf{p}$ for any n. We will assume without loss of generality that $p_1 = 0$. Then

$$\mathbf{p}M^n\mathbf{e}_1^T = \mathbf{p}\mathbf{e}_1^T = p_1 = 0.$$

So

$$p_2\mathbf{e}_2M^n\mathbf{e}_1^T + p_3\mathbf{e}_3M^n\mathbf{e}_1^T + \cdots + p_n\mathbf{e}_nM^n\mathbf{e}_1^T = 0.$$

However each of these terms are non-negative and so we have – in particular – that

$$p_i\mathbf{e}_iM^n\mathbf{e}_1^T = 0 \qquad \text{for } i = 2, 3, \ldots, k.$$

Not all of $p_2, \ldots, p_k$ can be zero (as they add to 1) and so there is at least one i such that it is impossible to get from the second state to the first state

$$\mathbf{e}_iM^n\mathbf{e}_1^T = 0.$$

It is then impossible to move from state i to state 1 in any finite time. It follows in an irreducible chain that each $p_i > 0$.

(ii) Say now that we have independent stationary distributions

$$\mathbf{p} = (p_1, \ldots, p_k) \quad \text{and} \quad \mathbf{P} = (P_1, \ldots, P_k).$$

If any of the p_i or P_i are zero then we know from (i) that the chain is not irreducible. So assume the p_i, P_i are all positive. Now for any real λ it follows that

$$(\lambda\mathbf{p} + (1-\lambda)\mathbf{P})M = \lambda\mathbf{p} + (1-\lambda)\mathbf{P}.$$

As $\mathbf{p}$ and $\mathbf{P}$ are independent then $\mathbf{p} \neq \mathbf{P}$. The vectors

$$\lambda\mathbf{p} + (1-\lambda)\mathbf{P}$$

constitute the line in $\mathbb{R}^k$ connecting $\mathbf{p}$ and $\mathbf{P}$. However the above vector will not have non-negative entries for all values of λ, though this will be the case for $0 \leqslant \lambda \leqslant 1$. We take λ_0 to be the smallest $\lambda_0 > 1$ such that one (or more) of its entries becomes 0. The vector

$$\lambda_0\mathbf{p} + (1-\lambda_0)\mathbf{P}$$

is still a probability vector with non-negative entries adding to 0, represents a stationary distribution, and one of its entries is zero. It follows then that the Markov chain is not irreducible.

(iii) The converse – that a unique stationary distribution implies irreducibility – is not true. For example, consider the two state Markov chain with transition matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the only stationary distribution is $(0, 1)$ but the chain is not irreducible as it is impossible to move out of the second state to the first state.