Solution (#1038) (i) The M for the given Markov chain equals

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix} = M$$

and so $M^n = M$ for all positive integers M.

The row vectors $\mathbf{x}$ which satisfy $\mathbf{x}M = \mathbf{x}$ are of the form $(a, 0, c)$. There isn't a unique probability vector $\mathbf{x}$ such that $\mathbf{x}M = \mathbf{x}$ as the long-term behaviour of the system depends solely on what happens on the first occasion. The state will either remain permanently in state A or in state C. If the initial probabilities were (a_0, b_0, c_0) then

$$a = a_0 + \frac{1}{2}b_0, \qquad c = \frac{1}{2}b_0 + c_0$$

are the probabilities of the system being permanently in states A or C.

(ii) The transition matrices P and Q equal

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note

$$P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad P^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad P^4 = I_4.$$

More generally

$$P^{4k} = I_4, \qquad P^{4k+1} = P, \qquad P^{4k+2} = P^2, \qquad P^{4k+3} = P^3.$$

As regards Q we have

$$Q^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Q^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $Q^k = Q^3$ for $k \geqslant 3$.