Nilpotent elements in Hochschild cohomology

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Abstract

We study the algebra $A = K\langle x, y\rangle/(x^2, y^2, (xy)^k + q(yx)^k)$ over the field K where $k \geq 1$, and where $0 \neq q \in K$. We determine a minimal projective bimodule resolution of A. In the case when q is not a root of unity, we compute its Hochschild cohomology. In particular we show that for $n \geq 3$, the n-th part $HH^n(A)$ has dimension $k-1$ if $\text{char}(K)$ does not divide k. We also show that every element in $HH^n(A)$ for $n \geq 1$ is nilpotent. This is motivated by the problem of understanding why the finite generation condition (Fg) fails which is needed to ensure existence of support varieties.

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To Dave

1 Introduction

Assume A is a finite-dimensional selfinjective algebra over some field. We would like to have support varieties for A-modules, similar to those for representations of finite groups. If G is a finite group then such varieties are constructed using group cohomology, which is graded commutative and noetherian. These varieties have very good properties and have proved to be a powerful tool in many contexts.

In [12] this is extended to finite-dimensional algebras, by defining support varieties based on Hochschild cohomology. It is shown in [6] that with suitable finiteness conditions, many of the properties in the group setting generalize.

Following the account of [13], the finiteness condition needed is called (Fg), and it states that the Hochschild cohomology $HH^*(A)$ should be noetherian, and the ext-algebra $\text{Ext}_A(A/J, A/J)$ of A should be finitely generated as a module over $HH^*(A)$, here J is the radical of A.

Here we study the local selfinjective algebras of dimension $4k$,

$$A = K\langle x, y\rangle/(x^2, y^2, (xy)^k + q(yx)^k)$$

where $k \geq 1$ and $0 \neq q \in K$. When $\text{char}(K) = 2$ and k is a power of 2 and moreover $q = -1$, this is isomorphic to the group algebra of a dihedral 2-group. It is known that the indecomposable non-projective modules are independent of q, and their parametrisation is independent of the characteristic (see 2.4). One would therefore expect that its homological algebra should be similar to the group situation.

However this is not the case. Take $k = 1$ and q not a root of unity, then the algebra is precisely the famous example of [4] which is selfinjective but its Hochschild cohomology is finite-dimensional. In particular condition (Fg) fails. For general k, there is a problem as well. Namely as it is shown in [8], when q is not a root of unity, there are A-modules with complexity one (that is, with bounded projective resolutions) which are not periodic. By Theorem 5.3 in [6] this implies that (Fg) cannot hold. One would like to understand why.

We determine explicitly a minimal projective bimodule resolution for A, for arbitrary characteristic, and arbitrary non-zero q. Furthermore, assume that q is not a root of unity. For such algebras, we compute the dimensions of the homogeneous parts of the Hochschild cohomology. We show that for $n \geq 3$, the n-th Hochschild cohomology $HH^n(A)$ has dimension $k-1$ if the characteristic of the field does not divide k, and otherwise it has dimension k. Moreover, we show that every element of positive degree is nilpotent. This uses a general result of [12] (see 2.3). In particular this shows that the Hochschild cohomology algebra is not noetherian for $k \geq 2$.

When $k = 1$ this shows again that the Hochschild cohomology is finite-dimensional, as in [4]: In this case the characteristic does not divide k and hence $HH^n(A) = 0$ for $n \geq 3$.

In general we see that the dimension of $HH^n(A)$ is bounded when q is not a root of unity. Given that the algebra A is of infinite representation type, this may be unexpected. It also is different from the group setting, see [10].

We describe now the content of this paper. The second section collects relevant background. In the third section we determine a minimal projective bimodule resolution, and in the fourth section we compute the cohomology.

2 Preliminaries

2.1 The algebras

The algebra A with $q = -1$ is symmetric, and it is one of the algebras of dihedral type introduced in [5]. In general, the socle of A is spanned by $(xy)^k$, and the factor algebra $A/\text{soc}(A)$ is independent of q. That is, the general algebra A is a socle deformation of an algebra of dihedral type.

Indecomposable non-projective A-modules are annihilated by $\text{soc}(A)$, and hence they are independent of q. However the action of the Heller operator Ω does depend on q: As mentioned above, for q not a root of unity there are modules of complexity one which are not Ω-periodic. On the other hand, when $q = -1$, all modules of complexity one have Ω-period ≤ 2.

Recall that if R is a finite-dimensional algebra then every finite-dimensional R-module M has a projective cover $\pi : P \to M$, and $\Omega(M)$ is the kernel of π. The kernels of the maps in a minimal projective resolution of M are the modules $\Omega^n(M)$ for $n \geq 1$, called syzygies. We use this when $R = A$ as above. As well, for Hochschild cohomology we use this when R is the enveloping algebra of A, that is $R = A^e = A \otimes_K A^{op}$, and when $M = A$. We write $\Omega_A(-)$ and we write just Ω for Ω_{A^e}.

2.2 Hochschild cohomology

The Hochschild cohomology $HH^*(A)$ is isomorphic to $\mathrm{Ext}^*_{A^e}(A,A)$. Here A is viewed as a left module for the enveloping algebra A^e. We will compute this from a minimal projective resolution of A as an A^e-module, equivalently, as $A-A-$bimodule:

$$(\mathbb{P}) \qquad \ldots \to \mathbb{P}_n \stackrel{d_n}{\to} \mathbb{P}_{n-1} \to \ldots \stackrel{d_1}{\to} \mathbb{P}_0 \stackrel{d_0}{\to} A \to 0$$

Here the kernel of d_{n-1} is the bimodule $\Omega^n(A)$. We identify $HH^n(A)$ with $\mathrm{Ext}^n_{A^e}(A,A)$, and to compute the dimension of $HH^n(A)$ we will use the exact sequence

$$0 \to \mathrm{Hom}_{A^e}(\Omega^{n-1}(A),A) \to \mathrm{Hom}_{A^e}(\mathbb{P}_{n-1},A) \to \mathrm{Hom}_{A^e}(\Omega^n(A),A) \to HH^n(A) \to 0.$$

The terms of a minimal projective bimodule resolution can be described more generally, by a result of Happel, see [9]. In our case, K is the only simple A-module, and A^e ($\cong A^e(1\otimes 1)$) is the only indecomposable projective A^e-module (up to isomorphism), so each $\mathbb{P}_n$ is isomorphic to a direct sum of copies of A^e. Happel's result becomes the following.

Proposition 2.1. *The multiplicity of A^e as a direct summand of $\mathbb{P}_n$ is equal to the dimension of $\mathrm{Ext}^n_A(K,K)$.*

In fact, tensoring $(\mathbb{P})$ over A with K on right gives a minimal projective resolution of K as a left A-module which is perhaps implicit in Happel's proof. The syzygies of K as a left A-module are well understood, we describe them below. This motivates the bimodule resolution we will construct in the next section.

2.2.1 The modules $\Omega^n_A(K)$

The algebra A is tame, and the indecomposable A-modules are classified. It is important for our context that the indecomposable non-projective modules are independent of q as explained above.

There is a parametrisation of indecomposables, originally due to C. M. Ringel [11]. The description can also be found in the appendix of [2] (the assumption on the characteristic and on k is not needed). The non-projective indecomposables come in two types, now called string modules and band modules, originally called modules 'of the first kind' and 'of the second kind'.

The syzygies of K are string modules. A string module for A can be visualized by a linear quiver where each vertex corresponds to a basis vector. Each arrow is labelled either by x or by y alternating, and the length of a maximal path with no change of orientation is bounded by $2k-1$. We describe $\Omega_A(K)$ and $\Omega^2_A(K)$ when $k=2$.

(a) The module $\Omega_A(K)$ may be written as:

$$Kv_1 \xrightarrow{x} Kv_2 \xrightarrow{y} Kv_3 \xrightarrow{x} Kv_4 \xleftarrow{y} Kv_5 \xleftarrow{x} Kv_6 \xleftarrow{y} Kv_7.$$

This means that x takes $v_1 \to v_2$, $v_3 \to v_4$ and $v_6 \to v_5$ and it maps all other basis vectors to zero. Moreover y acts by $v_2 \to v_3, v_7 \to v_6$ and $v_5 \to v_4$, and all other basis vectors are mapped to zero.

This module is generated by v_1, v_7 and can be identified with $\Omega_A(K)$, that is, $Ax + Ay \subset A$, by mapping $x \to v_7$ and $y \mapsto v_1$ which extends to an A-module isomorphism.

(b) $\Omega_A^2(K)$ is the following: We use shorthand notation, and write just v_i instead of Kv_i.

$$v_1 \xrightarrow{x} v_2 \xrightarrow{y} v_3 \xrightarrow{x} v_4 \xleftarrow{y} v_5 \xrightarrow{x} v_6 \xleftarrow{y} v_7 \xleftarrow{x} v_8 \xleftarrow{y} v_9.$$

One can turn this into an exact sequence of A-modules. The submodule generated by v_9 is isomorphic to Ax, and the submodule generated by v_1 is isomorphic to Ay. These two submodules have trivial intersection and span a maximal submodule. We have an exact sequence

$$0 \to Ax \oplus Ay \to \Omega_A^2(K) \to K \cong \Omega_A^0(K) \to 0.$$

(c) This generalizes. To obtain the description of $\Omega_A^{n+2}(K)$ one may start with $\Omega^n(K)$ and extend the quiver describing it at each end by a maximal linear subquiver of length $2k-1$. This reflects the fact that there is a short exact sequence of left A-modules

$$0 \to Ax \oplus Ay \to \Omega^{n+2}(K) \to \Omega^n(K) \to 0.$$

Since K is simple, we have for each n that

$$\mathrm{Ext}_A^n(K, K) \cong \mathrm{Hom}_A(\Omega^n(K), K).$$

The non-zero homomorphisms from $\Omega^n(K)$ map a vector corresponding to a sink to some scalar and map any other basis vector to zero. Hence we see that

$$\dim \mathrm{Ext}_A^n(K, K) = n + 1.$$

For the group setting, see also [1].

2.2.2 Small degrees

The first two terms of a minimal bimodule resolution can be described explicitly, more generally, for a finite-dimensional algebra of the form $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is a finite quiver and I is an admissible ideal of $K\mathcal{Q}$. Let $\mathcal{Q}_0$ be the set of vertices of the quiver, and $\mathcal{Q}_1$ be the set of arrows. If $\alpha : i \to j$ is an arrow, write $s\alpha = i$ and $t\alpha = j$. We have

$$\begin{aligned} \mathbb{P}_0 &= \oplus_{i \in \mathcal{Q}_0} A^e(e_i \otimes e_i) \\ \mathbb{P}_1 &= \oplus_{\alpha \in \mathcal{Q}_1} A^e(e_{s\alpha} \otimes_\alpha e_{t\alpha}). \end{aligned}$$

We must label the generators of $\mathbb{P}_1$ by arrows, so that we can distinguish equal idempotents in case of multiple arrows. The first two differentials are defined by

$$\begin{aligned} d_0(e_i \otimes e_i) &:= e_i \\ d_1(e_{s\alpha} \otimes_\alpha e_{t\alpha}) &:= \alpha(e_{t\alpha} \otimes e_{t\alpha}) - (e_{s\alpha} \otimes e_{s\alpha})\alpha, \end{aligned}$$

and the images of the generators of $\mathbb{P}_1$ labelled by arrows are minimal generators of $\Omega(A)$.

To describe minimal generators for $\Omega^2(A)$ the following notation is convenient.

Notation 2.2. (1) If $a_1a_2\dots a_m$ is a monomial in $K\mathcal{Q}$ then define

$$\rho(a_1\dots a_m) := \sum_{j=1}^{m} a_1a_2\dots a_{j-1}(e_{sa_j}\otimes_{a_j} e_{ta_j})a_{j+1}\dots a_m.$$

This is an element in $\mathbb{P}_1$, and ρ extends to a linear map $K\mathcal{Q}\to\mathbb{P}_1$.

(2) Then minimal generators for $\Omega^2(A)$ can be written down, roughly speaking, by the images of minimal relations under ρ.

(3) We return to the algebra A. Its quiver has one vertex, with idempotent 1_A, and two loops, x and y, and minimal relations x^2, y^2, $(xy)^k + q(yx)^k$. We label the generator $1\otimes 1$ of $\mathbb{P}_0$ as $[f_0^0]$. The generators of $\mathbb{P}_1$ are written as $[f_0^1]$, corresponding to x, and $[f_1^1]$, corresponding to y. Note that as elements of A^e, each of these is $1\otimes 1$. With this,

$$d_1([f_0^1]) = x\otimes 1 - 1\otimes x = x[f_0^0] - [f_0^0]x =: f_0^1$$

and

$$d_1([f_1^1]) = y\otimes 1 - 1\otimes y = y[f_0^0] - [f_0^0]y =: f_1^1.$$

(4) The module $\ker(d_1) = \Omega^2(A)$ has generators $\rho(x^2)$ and $\rho(y^2)$ and $\rho((xy)^k+q(yx)^k)$. Here

$$f_0^2 := \rho(x^2) = x[f_0^1] + [f_0^1]x \quad \text{and} \quad f_2^2 := \rho(y^2) = y[f_1^1] + [f_1^1]y.$$

We will introduce a shorthand notation for the element $\rho((xy)^k + q(yx)^k)$ of $\mathbb{P}_1$.

We define the following linear maps: If M is an $A-A-$bimodule, define a linear map $Tr_{xy} : M \to M$ which substitutes for each occurence of x, and a linear map $Tr_{yx} : M\to M$ which substitutes for each occurence of y,

$$Tr_{xy}(m) := \sum_{t=0}^{k-1}(xy)^t(m)(yx)^{k-1-t}$$

$$Tr_{yx}(m) := \sum_{t=0}^{k-1}(yx)^t(m)(xy)^{k-1-t}.$$

With this, we have the formula

$$\rho((xy)^k + q(yx)^k) = (Tr_{xy}[f_0^1])y + qy(Tr_{xy}[f_0^1]) + x(Tr_{yx}[f_1^1]) + q(Tr_{yx}[f_1^1])x.$$

The maps Tr_{xy} and Tr_{yx} will appear again later.

2.3 Nilpotent elements

We wish to identify nilpotent elements of Hochschild cohomology. This can sometimes be done without knowing the algebra structure of $HH^*(A)$ explicitly, by exploiting a more general result, that is Proposition 4.4 of [12].

Proposition 2.3. *Assume K is a field and A is a finite-dimensional K-algebra with radical J.*
Let $\eta\in HH^n(A)$ such that $\eta\otimes_A A/J$ is zero in $\mathrm{Ext}_A^n(A/J, A/J)$. Then η is nilpotent in $HH^*(A)$ with nilpotency index at most the radical length of A.

2.4 Independence of q

The indecomposable non-projective A-modules are independent of q. This holds because the socle of A (spanned by $(xy)^k$) annihilates any indecomposable non-projective module. This is more general, we give the argument as it might be useful elsewhere, and we could not find it in the literature.

Lemma 2.4. *Assume A is a finite-dimensional selfinjective K-algebra. If M is indecomposable and not projective (hence not injective) then the socle of A annihilates M.*

Proof Assume the socle of A does not annihilate M. Then there is a primitive idempotent e_i of A such that $\mathrm{soc}(Ae_i)M \neq 0$. Choose and fix $m \in M$ such that $\omega m \neq 0$ for $\omega = \omega e_i \in \mathrm{soc} Ae_i$. Then $A\omega = \mathrm{soc}(Ae_i)$ since $\mathrm{soc}(Ae_i)$ is simple. We have an A-module homomorphism $\varphi : Ae_i \to M$ defined by $\varphi(ae_i) = ae_i m$. Then $\varphi(\mathrm{soc}(Ae_i)) = \varphi(A\omega) = A\omega m \neq 0$. Hence the kernel of φ is zero, and φ is a split monomorphism since Ae_i is also injective. It follows that $M \cong Ae_i$ and M is injective and projective.

3 A minimal bimodule resolution

3.1 From now we assume that A is a local algebra of dihedral type as above. In this section we will construct a minimal resolution of A as an A^e-module. Here the parameter q is arbitrary, as well the field is arbitrary. As we have seen, the n-th term of a minimal bimodule resolution is free of rank $n+1$.

Our aim is to find recursively generators for the kernel of $\Omega^n(A)$. By the above there will be $n+1$ generators, and we will write these as $f_0^n, \ldots, f_n^n$. Then we label generators of $\mathbb{P}_n$ as

$$[f_0^n], \ldots, [f_n^n]$$

(each of these is equal to $1 \otimes 1$ as an element in A^e but of course we need to distinguish different copies). With this, we define the differential $d_n : \mathbb{P}_n \to \mathbb{P}_{n-1}$ by

$$[f_i^n] \mapsto f_i^n \in \mathbb{P}_{n-1}.$$

We will show that the f_i^n are in the kernel of d_{n-1}. To see that they are generators, there are several general arguments in the literature. In this case, one can even see directly that if U_n is the sub-bimodule generated by the f_i^n then $U_n \otimes_A K \cong \Omega_A^n(K)$.

In Notation 2.2, we have already fixed the generators of $\mathbb{P}_0$, the differential d_0, generators for $\Omega(A)$, the differential d_1 and generators for $\Omega^2(A)$.

3.2 Before giving the general definition, we will explain why $\Omega^r(A) \otimes_A K$ is isomorphic to $\Omega_A^r(K)$ for $r = 1, 2$. Since $(-) \otimes_A K$ annihilates the radical, we see

$$A(f_0^1) \otimes_A K = Ax(1 \otimes 1) \otimes_A K \cong Ax$$

and similarly $A(f_1^1) \otimes_A K \cong Ay$ and we see directly that $\mathrm{Ker}(d_0) \otimes_A K \cong Ax + Ay = \Omega_A(K)$.

Similarly consider $\mathrm{Ker}(d_1) \otimes_A K$, this has submodules

$$A(f_0^2) \otimes_A K \cong Ax, \quad \text{and} \quad A(f_2^2) \otimes_A K \cong Ay.$$

Consider $\zeta := f_1^2 \otimes_A K$. This has only two terms, namely

$$\zeta = (xy)^{k-1}x[f_1^1] \otimes_A K + q(yx)^{k-1}y[f_0^1] \otimes_A K.$$

This generates a 3-dimensional left A-submodule of $\mathbb{P}_1$, isomorphic to A/J^2. We can also see that the intersection of $A\zeta$ with $A(f_0^2) \otimes_A K$ is equal to $y\zeta$, and the intersection with $A(f_2^2) \otimes_A K$ is equal to $x\zeta$, both one-dimensional. We see that $\Omega^2(A) \otimes_A K \cong \Omega_A^2(K)$.

The formulae below for generators may similarly be thought of as being a lift of a one-sided minimal projective resolution.

3.3 We will inductively define explicit minimal generators of $\Omega^n(A)$, they will be denoted by f_i^n for $0 \leq i \leq n$, they are elements of $\mathbb{P}_{n-1}$. Then we fix generators of $\mathbb{P}_n$ and label them by $[f_i^n]$ for $0 \leq i \leq n$. With this, we define the differential $d_n : \mathbb{P}_n \to \mathbb{P}_{n-1}$

$$d_n : \; [f_i^n] \longrightarrow f_i^n \quad (0 \leq i \leq n).$$

Note that the elements f_i^n will be expressed in terms of $[f_i^{n-1}]$.

3.3.1 For any $n \geq 1$, we may set

$$f_0^n := x[f_0^{n-1}] + (-1)^n[f_0^{n-1}]x, \quad f_n^n := y[f_{n-1}^{n-1}] + (-1)^n[f_{n-1}^{n-1}]y.$$

We can see directly, using the recipe for the differentials, that these elements are in the kernel of d_{n-1}.

These may be thought of a lift to A^e of the process of 'extending by Ax and Ay'. When $n = 1$, these two elements generate $\Omega(A)$. We have also already written down generators for $\Omega^2(A)$.

3.3.2 We write down the remaining generators for $\Omega^3(A)$, the kernel of $d_2 : \mathbb{P}_2 \to \mathbb{P}_1$.

$$f_1^3 := ([f_0^2](yx)^{k-1}y - q^2(yx)^{k-1}y[f_0^2]) \;-\; (x[f_1^2] - q[f_1^2]x)$$

$$f_2^3 = ([f_1^2]y - qy[f_1^2]) \;-\; ((xy)^{k-1}x[f_2^2] - q^2[f_2^2](xy)^{k-1}x).$$

It is straightforward to check that these are in the kernel of d_2. By Proposition 2.1 we know that $\Omega^3(A)$ has four independent generators, and then by general arguments or directly it follows that the f_i^3 for $0 \leq i \leq 3$ generate $\Omega^3(A)$.

Now we state the answer for degrees $n \geq 4$. Here the field is arbitrary, and q can be any non-zero element of K.

Proposition 3.1. *Assume $n \geq 4$. Then generators for $\Omega^n(A)$ may be taken as follows:*
(a) Let $1 \leq i < (n/2)$. Then

$$f_i^n = ([f_{i-1}^{n-1}](yx)^{k-1}y + (-1)^n q^{n-i}(yx)^{k-1}y[f_{i-1}^{n-1}]) + (-1)^i(x[f_i^{n-1}] + (-1)^n q^i[f_i^{n-1}]x).$$

(b) If n is even and $i = (n/2)$ then we have a generator f_i^n equal to

$$f_i^n = (Tr_{xy}[f_{i-1}^{n-1}])y + q^i(yTr_{xy}[f_{i-1}^{n-1}]) + (-1)^i x(Tr_{yx}[f_i^{n-1}]) + (-1)^i q^i(Tr_{yx}[f_i^{n-1}])x$$

with the notation as in 2.2.
(c) If $(n/2) < i \leq n-1$ then

$$f_i^n = ([f_{i-1}^{n-1}]y + (-1)^n q^{n-i}y[f_{i-1}^{n-1}]) \ + \ (-1)^i((xy)^{k-1}x[f_i^{n-1}] + (-1)^n q^i[f_i^{n-1}](xy)^{k-1}x)$$

(d) For any n, we have

$$f_0^n = x[f_n^{n-1}] + (-1)^n[f_0^{n-1}]x, \quad f_n^n = y[f_{n-1}^{n-1}] + (-1)^n[f_{n-1}^{n-1}]y.$$

Proof Part (d) has already been explained above. Parts (a) to (c) are proved by induction. We give details for part (b) and leave the proof for (a) and (c) to the reader, it is fairly straightforward.
Assume $i = n/2$ and $n \geq 4$, and we have the formulae for degree $n-1$ as the inductive hypothesis.
We take the stated expression f_i^n, and we must show that d_{n-1} maps this to zero. Recall that d_{n-1} is obtained by 'removing brackets', that is, by substituting f_{i-1}^{n-1} and f_i^{n-1} into the relevant terms.

We first substitute f_{i-1}^{n-1}, which occurs in the first two terms of (b). Since $i-1 < (n-1)/2$ we must use the formula from part (a), that is

$$f_{i-1}^{n-1} = [f_{i-2}^{n-2}](yx)^{k-1}y + (-1)^{n-1}q^{n-i}(yx)^{k-1}y[f_{i-2}^{n-2}] + (-1)^{i-1}x[f_{i-1}^{n-2}] + (-1)^{n-i}q^{i-1}[f_{i-1}^{n-2}]x.$$

(1) We substitute $[f_{i-2}^{n-2}](yx)^{k-1}y$ into f_i^n. Since $y^2 = 0$, all terms from f_i^n with a factor $[f_{i-1}^{n-1}]y$ become zero. This leaves only one term, namely

$$q^i(yx)^{k-1}y[f_{i-2}^{n-2}](yx)^{k-1}y.$$

(2) We substitute $(-1)^{n-1}q^{n-i}(yx)^{k-1}y[f_{i-2}^{n-2}]$. Similarly almost all terms are zero, and we are left with

$$(-1)^{n-1}q^{n-i}(yx)^{k-1}y[f_{i-2}^{n-2}](yx)^{k-1}y.$$

Since $i = n/2$ we have $n - i = i$ and also $(-1)^{n-1} = (-1)$ and this term cancels against the term in (1).

(3) Substituting $(-1)^{i-1}x[f_{i-1}^{n-2}] + (-1)^{n-i}q^{i-1}[f_{i-1}^{n-2}]x$ into f_i^n gives the four terms

$$(-1)^{i-1}Tr_{xy}(x[f_{i-1}^{n-2}])y + (-1)^{i-1}q^i yTr_{xy}(x[f_{i-1}^{n-2}])$$

$$+(-1)^{n-i}q^{i-1}(Tr_{xy}(f_{i-1}^{n-2}]x)y + (-1)^{n-i}q^{2i-1}y(Tr_{xy}([f_{i-1}^{n-2}]x).$$

(4) Now we apply d_{n-1} to the terms of f_i^n with $[f_i^{n-1}]$. We substitute using the formula from part (c), note that $i > (n-1)/2$. This has two terms with $[f_i^{n-2}]$. We substitute these, and exactly as in (1) and (2), most of these are zero and the two terms left cancel out.

(5) We substitute the contribution

$$[f_{i-1}^{n-2}]y + (-1)^{n-1}q^{n-1-i}y[f_{i-1}^{n-2}]$$

into the last two terms of f_i^n. These are four terms, namely

$$\begin{aligned}&(-1)^i xTr_{yx}([f_{i-1}^{n-2}]y) + (-1)^i q^i (Tr_{yx}[f_{i-1}^{n-2}]y)x \\ &+ (-1)^{i+n-1}q^{n-1-i}x(Tr_{yx}(y[f_{i-1}^{n-2}]) + (-1)^{i+n-1}q^{n-1}(Tr_{yx}(y[f_{i-1}^{n-2}])x.\end{aligned}$$

(6) We compare (3) and (5). The first term of (5) cancels against the first term of (3) since $x(Tr_{yx}[f_{i-1}^{n-2}]y) = Tr_{xy}(x[f_{i-1}^{n-2}])y$. Similarly the last term of (5) cancels agains the last term of (3).

Now consider the second term (3), which is

$$(-1)^{i-1}q^i\sum_{t=0}^{k-1}(yx)^{t+1}[f_{i-1}^{n-2}](yx)^{k-1-t}.$$

We add to this the second term of (5). All but two terms cancel and we are left with

$$(*) \qquad (-1)^{i-1}q^i(yx)^k[f_{i-1}^{n-2}] + (-1)^i q^i [f_{i-1}^{n-2}](yx)^k.$$

Similarly adding the third term of (3) and the third term of (5) has just two terms left,

$$(**) \qquad (-1)^{n-i}q^{i-1}[f_{i-1}^{n-2}](xy)^k + (-1)^{n+i-1}q^{i-1}(xy)^k[f_{i-1}^{n-2}].$$

In (*), we substitute $(-q)(yx)^k = (xy)^k$, and adding (*) and (**) gives zero, as required. □

4 Homomorphisms and $HH^n(A)$

We assume from now that q is not a root of unity. The aim is to prove the following.

Theorem 4.1. *The dimension of $HH^n(A)$ is*

$$\begin{cases} k+1 & n=1 \\ k & n=2 \\ k & n\geq 3, \quad \text{char}(K)\ |k \\ k-1 & n\geq 3, \quad \textit{else.}\end{cases}$$

That is, as a bimodule, A has bounded cohomology. When $k=1$, this recovers the result of [4]. On the way we will also see the following.

Theorem 4.2. *Let $n \geq 2$. If $\varphi : \Omega^n(A) \to A$ is a homomorphism then the image of φ is contained in the radical of A. Hence the class of φ in $HH^n(A)$ is nilpotent.*

To prove Theorem 4.1, we determine the dimensions of the relevant spaces of homomorphisms.

Proposition 4.3. *Let $r_n := \dim \mathrm{Hom}_{A^e}(\Omega^n(A), A)$ for $n \geq 0$.*
We have $r_0 = k+1$ and $r_1 = 4k$; and
(i) If $n = 2t \geq 2$ then $r_n = 2kn + k$.
(ii) If $n = 2t+1 \geq 3$ then

$$r_n = \begin{cases} 2kn + 2k & \mathrm{char}(K)|k \\ 2kn + (2k-1) & \text{else} \end{cases}$$

4.1 Fix $n \geq 1$. We may identify

$$\mathrm{Hom}_{A^e}(\Omega^n(A), A) = \{\varphi : \mathbb{P}_n \to A \mid \varphi(\Omega^{n+1}(A)) = 0\}$$

Recall that $\mathbb{P}_n$ is the projective A^e-module with generators $[f_j^n]$ for $0 \leq j \leq n$. A homomorphism from $\mathbb{P}_n$ to A is therefore determined by the images

$$\zeta_j := \varphi([f_j^n]) \ (0 \leq j \leq n).$$

The submodule $\Omega^{n+1}(A)$ of $\mathbb{P}_n$ is generated by the f_i^{n+1} for $0 \leq i \leq n+1$ as defined in Proposition 3.1 for $n+1 \geq 4$ and for $n = 1, 2$ in Notation 2.2 and 3.3. To find the homomorphisms we must determine precisely the ζ_j such that $\varphi(f_i^{n+1}) = 0$ for all i, that is, where we get zero if we substitute the ζ_j into the formula for f_i^{n+1}. Therefore we must solve the following system of equations for ζ_j:

We start with n such that $n+1 \geq 4$, then by Proposition 3.1 the equations are

$$(E_0) \quad x\zeta_0 + (-1)^{n+1}\zeta_0 x = 0 \quad \text{and} \quad (E_{n+1}) \quad \zeta_n y + (-1)^{n+1} y\zeta_n = 0$$

$$(E_i) \ \ \zeta_{i-1}(yx)^{k-1}y + (-1)^{n+1}q^{n+1-i}(yx)^{k-1}y\zeta_{i-1} = (-1)^{i+1}(x\zeta_i + (-1)^{n+1}q^i\zeta_i x) \ (1 \leq i < (n+1)/2)$$

$$(E_i) \ \ \zeta_{i-1}y + (-1)^{n+1}q^{n+1-i}y\zeta_{i-1} = (-1)^{i+1}[(xy)^{k-1}x\zeta_i + (-1)^{n+1}q^i\zeta_i(xy)^{k-1}x] \ ((n+1)/2 < i \leq n)$$

and if $n+1$ is even, also $i = (n+1)/2$ and

$$(E_{(n+1)/2}) \quad Tr_{xy}(\zeta_{i-1})y + q^i y Tr_{xy}(\zeta_{i-1}) = (-1)^{i+1}(xTr_{yx}(\zeta_i + q^i(Tr_{yx}(\zeta_i)x))$$

The form of these identities shows that we must understand the spaces

$$\mathcal{X}_\alpha := \{\eta \in A : x\eta + \alpha\eta x = 0\}, \quad \text{and} \ \ \mathcal{Y}_\alpha := \{\eta \in A : \eta y + \alpha y\eta = 0\}$$

for $0 \neq \alpha \in K$. One checks that

$$\mathcal{X}_\alpha = \begin{cases} \mathrm{Sp}\{1, x\} + \widehat{\mathcal{X}}_\alpha & \alpha = -1 \\ \mathrm{Sp}\{x, (yx)^{k-1}y\} + \widehat{\mathcal{X}}_\alpha & \alpha = q \\ \mathrm{Sp}\{x\} + \widehat{\mathcal{X}}_\alpha & \text{else} \end{cases}$$

$$\mathcal{Y}_\alpha = \begin{cases} \mathrm{Sp}\{1, y\} + \widehat{\mathcal{X}}_\alpha & \alpha = -1 \\ \mathrm{Sp}\{y, (xy)^{k-1}x\} + \widehat{\mathcal{Y}}_\alpha & \alpha = q \\ \mathrm{Sp}\{y\} + \widehat{\mathcal{Y}}_\alpha & \text{else} \end{cases}$$

where

$$\widehat{\mathcal{X}}_\alpha := \mathrm{Sp}\{(xy)^i - \alpha(yx)^i : 1 \le i \le k-1\} \cup \{(xy)^i x, 1 \le i \le k-1\} \cup \{(xy)^k\}$$

$$\widehat{\mathcal{Y}}_\alpha := \mathrm{Sp}\{(xy)^i - \alpha(yx)^i : 1 \le i \le k-1\} \cup \{(yx)^i y, 1 \le i \le k-1\} \cup \{(xy)^k\}$$

and $\widehat{\mathcal{X}}_\alpha$ and $\widehat{\mathcal{Y}}_\alpha$ each has dimension $2k-1$.

4.1 The proof of 4.3 for $n+1 \ge 4$

Assume $n+1 \ge 4$, where $n = 2t$ or $n = 2t+1$. We will prove the following:
(1) $\zeta_0 = b_0 x + \widehat{\zeta_0}$ with $\widehat{\zeta_0} \in \widehat{\mathcal{X}}_{(-1)^{n+1}}$ and $b_0 \in K$. Moreover $b_0 = 0$ if $(-1)^n = 1$.
(2) For $1 \le i \le t$ we have $\zeta_i = b_1 x + c_i (yx)^{k-1} y + \widehat{\zeta_i}$ where $\widehat{\zeta_1} \in \widehat{\mathcal{X}}_{(-1)^{n+1}q^i}$ and $b_i, c_i \in K$ and c_i is a function of b_{i-1}, except that if $(-1)^n = 1$ then c_1 is arbitrary.

(3) $\zeta_n = b_n y + \widehat{\zeta_n}$ with $\widehat{\zeta_n} \in \widehat{\mathcal{Y}}_{(-1)^{n+1}}$ and $b_n \in K$. If $(-1)^n = 1$ then $b_n = 0$.
(4) For $t+1 \le i \le n-1$, we have $\zeta_i = b_i y + c_i (xy)^{k-1} x + \widehat{\zeta_i}$ with $\widehat{\zeta_i} \in \widehat{\mathcal{Y}}_{(-1)^{n+1}q^i}$ and $b_i, c_i \in K$ and where c_i is a function of b_{i+1}; except if $(-1)^n = 1$ then c_{n-1} is arbitrary.

When $n = 2t$ there is an additional condition on ζ_t, and when $n = 2t+1$ we may have an extra condition relating ζ_t and ζ_{t+1}:

(5) Suppose $n = 2t$. Then $\zeta_t = c_t (yx)^{k-1} y + c_t' (xy)^{k-1} x + \widetilde{\zeta_t}$ with $\widehat{\zeta_t} \in \widehat{\mathcal{X}}_{-q^t} \cap \widehat{\mathcal{Y}}_{-q^t}$ and $c_t, c_t' \in K$ where c_t is a function of b_{t-1} and c_t' is a function of b_{t+1}.
(6) Suppose $n = 2t+1$. If $\mathrm{char}(K)$ does not divide k then $b_t = (-1)^t b_{t+1}$. Otherwise b_t, b_{t+1} are arbitrary.

Proof One checks that the elements listed satisfy the identities. We show that these are all.
(1) For identity (E_0) we require $\zeta_0 \in \mathcal{X}_{(-1)^{n+1}}$, and hence $\zeta_0 = a_0 + b_0 x + \widehat{\zeta_0}$ with $a_0, b_0 \in K$ and $\widehat{\zeta_0} \in \widehat{\mathcal{X}}_{-1}$. We substitute into (E_1) and get

$$a_0(1 - q^n)(yx)^{k-1} y + b_0(1 + q^{n-1})(xy)^k = x\zeta_1 + (-1)^{n+1} q \zeta_1 x.$$

The element $(yx)^{k-1}y$ is not of the form $x\eta + q\eta x$ for $\eta \in A$ and hence $a_0 = 0$, and the claim holds for ζ_0. Moreover, the element $(xy)^k$ is not of the form $x\eta + q\eta x$ for $\eta \in A$. So if $(-1)^{n+1} = 1$ then also $b_0 = 0$.
If $(-1)^{n+1} = 1$ then $\zeta_1 \in \mathcal{X}_{(-1)^{n+1}q}$ and we may write $\zeta_1 = b_1 x + c_1 (yx)^{k-1} Y \widehat{\zeta_1}$ with $b_1, c_1 \in K$ arbitrary, and $\widehat{\zeta_1} \in \widehat{\mathcal{X}}_{(-1)^{n+1}q}$.
Suppose $(-1)^{n+1} \ne 1$, then the identity (E_1) is satisfied for $\zeta_1 = c_1 (yx)^{k-1} y + \zeta_1'$ where c_1 is a function of b_0, and where $x\zeta_1' + (-1)^{n+1} q \zeta_1' x = 0$. That is $\zeta_1' = b_1 x + \widehat{\zeta_1}$ and $\widehat{\zeta_1} \in \widehat{\mathcal{X}}_{(-1)^{n+1}q}$. This proves the claim for ζ_1.

(2) The case $i=1$ is done, and we continue by induction: Assume true for $i-1$, and $i<t$. Then we substitute ζ_{i-1} into (E_i) and get

$$b_{i-1}(1+(-1)^n q^{n-i})(xy)^k = (-1)^{i+1}(x\zeta_i + (-1)^{n+1}q^i\zeta_i x).$$

This is satisfied with $\zeta_i = c_i(yx)^{k-1}y + \zeta_i'$ where $c_i \in K$ is a function of b_{i-1} and where $x\zeta_i' + (-1)^{n+1}q^i\zeta_i' x = 0$. Then

$$\zeta_i' = b_i x + \widehat{\zeta}_i$$

and $b_i \in K$ and $\widehat{\zeta}_i \in \widehat{\mathcal{X}}_{(-1)^{n+1}q^i}$.

(3) For (E_{n+1}) we require $\zeta_n \in \mathcal{Y}_{(-1)^{n+1}}$ and hence $\zeta_n = a_n + b_n y + \widehat{\zeta}_{(-1)^{n+1}}$. We substitute into identity (E_n) and get

$$\zeta_{n-1}y+(-1)^{n+1}q\zeta_{n-1}y = (-1)^{n+1}[a_n(1+(-1)^{n+1}q^n)(xy)^{k-1}x+b_n(1+(-1)^n q^{n-1})(xy)^k].$$

As in (1) we deduce $a_n = 0$. If $(-1)^n = 1$ then $\zeta_n \in \mathcal{Y}_{(-1)^{n+1}q}$ and $\zeta_{n-1} = b_{n-1}x + c_{n-1}(yx)^{k-1}y + \widehat{\zeta}_{n-1}$ with b_{n-1} and $c_{n-1} \in K$ and $\widehat{\zeta}_{n-1} \in \widehat{\mathcal{Y}}_{(-1)^{n+1}q}$. Otherwise we get the same description but now c_{n-1} is a function of b_{n-1}.
(4) By downwards induction, one obtains the stated expression for ζ_i from (E_{i+1}) for $t+1 \le i \le n$.
(5) Assume $n = 2t$. Then we have an expression for ζ_t from the last step of (4). We also have an expression for ζ_t from the last step of (3). These must be the same, and we deduce that it has no terms with x, y and that it is of the stated form.
(6) Assume $n = 2t+1$. Then by (1) to (4) we have expressions for all ζ_i, and all identities (E_j) for $j \neq t+1$ are satisfied.
We substitute ζ_t and ζ_{t+1} into (E_{t+1}). This gives

$$b_t[Tr_{xy}(x)y + q^{t+1}yTr_{xy}(x)] \;=\; (-1)^{t+2}b_{t+1}[xTr_{yx}(y) + q^{t+1}Tr_{yx}(y)x].$$

That is

$$kb_t(1-q^t)(xy)^k = (-1)^t kb_{t+1}(1-q^t)(xy)^k.$$

If char(K) does not divide k, we need $b_t = (-1)^t b_{t+1}$. Otherwise b_t, b_{t+1} are arbitrary.

We compute now the dimension to complete the proof of Propositon 4.3. We have $n = 2t$ or $2t+1$ and $n \ge 4$.
(1) Assume n is even. For each $i \neq t$ with $0 \le i \le n$ we have the summand $\widehat{\zeta}_i$ which can be arbitrary in a space of dimension $2k-1$. We have also $\widetilde{\zeta}_t$ in a space of dimension k, the intersection of $\widehat{\mathcal{X}}_{-q^t} \cap \widehat{\mathcal{Y}}_{-q^t}$. Moreover we have n independent scalar parameters (in all cases). This gives in total

$$r_n = n(2k-1) + k + n = 2kn + k,$$

as stated.
(2) Now assume n is odd. Then for each i with $0 \le i \le n$ we have the summand $\widehat{\zeta}_i$ which can be arbitrary in a space of dimension $2k-1$. Moreover we have $n+1$ independent scalars if char(K) divides k and n independent scalars otherwise. So the dimension is either $r_n = (n+1)(2k-1) + (n+1) = (n+1)2k$ or is $(n+1)(2k) - 1$.

4.2 Small cases

4.2.0 Let $n = 0$. The dimension $r_0 = \dim Z(A)$ is equal to the dimension of $\mathcal{X}_{-1} \cap \mathcal{Y}_{-1}$ which is equal to $k+1$, see 4.1.

4.2.1 We find r_1. Let $\varphi : \mathbb{P}_1 \to A$ be a homomorphism such that $\varphi(\Omega^2(A)) = 0$. Then φ is determined by elements $\varphi([f_i^1]) = \zeta_i$ in A, for $i = 0, 1$. We have $\varphi(\Omega^2(A)) = 0$ if and only if the following identities (E_i) hold, obtained from Notation 2.2:

$$(E_0)\ \ x\zeta_0 + \zeta_0 x = 0 \quad \text{and} \quad (E_2)\ \ \zeta_1 y + y\zeta_1 = 0$$

$$(E_1)\ \ Tr_{xy}(\zeta_0)y + qyTr_{xy}(\zeta_0) = -(xTr_{yx}(\zeta_1) + qTr(\zeta_1)x).$$

We will show that these are satisfied if and only if
(1) $\zeta_0 = b_0 x + \widehat{\zeta_0}$ with $b_0 \in K$ and $\widehat{\zeta_0} \in \widehat{\mathcal{X}_1}$ and
(2) $\zeta_1 = b_1 y + \widehat{\zeta_1}$ with $b_1 \in K$ and $\widehat{\zeta_1} \in \widehat{\mathcal{Y}_1}$.

These satisfy the (E_i) and we show that there are no others. By (E_0) we may write $\zeta_0 = a_0 + b_0 x + \widehat{\zeta_0}$ where $a_0, b_0 \in K$ and where $\widehat{\zeta_0} \in \widehat{\mathcal{X}_1}$. Similarly by (E_2) we have $\zeta_1 = a_1 + b_1 y + \widehat{\zeta_1}$ with $a_1, b_1 \in K$ and where $\widehat{\zeta_1} \in \widehat{\mathcal{Y}_1}$. We substitute ζ_0 and ζ_1 into (E_1), the terms with $\widehat{\zeta_i}$ become zero. The terms with b_0 and b_1 satisfy

$$kb_0((xy)^k + q(yx)^k) = -[kb_1((xy)^k + q(yx)^k)],$$

namely both sides are zero because of the relation. The terms with with a_0 and a_1 must satisfy

$$a_0(1+q)(yx)^{k-1}y = -[a_1(1+q)(xy)^{k-1}y],$$

and hence $a_0 = 0 = a_1$.
With (1) and (2) we have

$$r_1 = \dim \widehat{\mathcal{X}_1} + \dim \widehat{\mathcal{Y}_1} + 2 = 2(2k-1) + 2 = 4k.$$

4.2.2 We find r_2. Consider a homomorphism $\varphi : \mathbb{P}_2 \to A$ which maps $\Omega^3(A)$ to zero. The homomorphism φ is determined by $\varphi[f_i^2] = \zeta_i \in A$ for $0 \le i \le 2$. Then $\varphi(\Omega^3(A)) = 0$ if and only if the following identities (E_i) hold, obtained from 3.3:

$$(E_0)\quad x\zeta_0 - \zeta_0 x = 0 \quad \text{and} \quad (E_3)\quad \zeta_2 y - y\zeta_2 = 0;$$

$$\begin{aligned}(E_1)\quad \zeta_0(yx)^{k-1}y - q^2(yx)^{k-1}y\zeta_0 \ &= \ x\zeta_1 - q\zeta_1 x;\\ (E_2)\quad \zeta_1 y - qy\zeta_1 \ &= \ (xy)^{k-1}x\zeta_2 - q^2\zeta_2(xy)^{k-1}x.\end{aligned}$$

Precisely as in the proof for the general case, one shows that these hold if and only if
(1) $\zeta_0 = b_0 x + \widehat{\zeta_0}$ with $b_0 \in K$ and $\widehat{\zeta_0} \in \widehat{\mathcal{X}}_{-1}$, and $b_0 = 0$ if $\mathrm{char}(K) = 2$.
(2) $\zeta_2 = b_2 y + \widehat{\zeta_2}$ with $b_2 \in K$ and $\widehat{\zeta_2} \in \widehat{\mathcal{Y}}_{-1}$, and $b_2 = 0$ if $\mathrm{char}(K) = 2$.
(3) $\zeta_1 = c_1(yx)^{k-1}y + c_1'(xy)^{k-1}x + \widetilde{\zeta_1}$ with $\widetilde{\zeta_1} \in \widehat{\mathcal{X}}_{-q} \cap \widehat{\mathcal{Y}}_{-q}$, and where $c_1, c_1' \in K$. If $\mathrm{char}(K) \neq 2$ then c_1 is a function of b_0 and c_1' is a function of b_2. If $\mathrm{char}(K) = 2$ then c_1, c_1' are arbitrary.

With this, we note that the number of independent scalar parameters is the same whether or not the characteristic is 2, and we compute that

$$r_2 = \dim \widehat{\mathcal{X}}_{-1} + \dim \widehat{\mathcal{Y}}_{-1} + 2 + \dim(\widehat{\mathcal{X}}_{-q} \cap \widehat{\mathcal{Y}}_{-q}) = 4kn + k.$$

We prove Theorem 4.1, using the exact sequence

$$0 \to \mathrm{Hom}_{A^e}(\Omega^{n-1}(A), A) \to \mathrm{Hom}_{A^e}(\mathbb{P}_{n-1}, A) \to \mathrm{Hom}_{A^e}(\Omega^n(A), A) \to HH^n(A) \to 0.$$

The dimension of $\mathrm{Hom}_{A^e}(\mathbb{P}_{n-1}, A)$ is $4kn$ and hence

$$\dim HH^n(A) = r_n + r_{n-1} - 4kn$$

and we get the stated answer.

We prove Theorem 4.2. The description of the most general homomorphism $\Omega^n(A) \to A$ in 4.3 and in the small cases shows that its image is contained in the radical of A. Hence by Proposition 2.3 every element of positive degree is in $HH^*(A)$ is nilpotent.

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