Decomposition numbers for symmetric groups

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Representations of symmetric groups

$G = \mathcal{S}_n$, V = finite-dimensional K-vector space.

Representation = group homomorphism $\rho : G \to GL(V)$.

V = G-module [$vg := (v)(g\rho)$, $g \in G, v \in V$.]

$\Omega^{\{2\}} =$ 2-element subsets of $\{1, 2, \ldots, n\}$.

$K = \mathbb{Z}_2$.

$K\Omega^{\{2\}} = M^{(n-2,2)}$ permutation module of $\mathcal{S}_n$.

$\{i, j\}g = \{(i)g, (j)g\}$

Qu. Composition factors? Same for $M^{(n-3,3)}$?

Qu. Same for eg $M^{(n-5,3,2)}$?

Specht modules

λ partition of n, S^{λ} := Specht module.

- characteristic-free.
- explicit: submodule of permutation module.

Eg $S^{(n)}$ = the trivial module.

$\Omega = \{1, 2, \ldots, n\}$, $K\Omega = \mathrm{Span}\{v_i\} \cong M^{(n-1,1)}$.

$$S^{(n-1,1)} \cong \{\sum_i c_i v_i : \sum_i c_i = 0\} \subset K\Omega.$$

- $K = \mathbb{C}$: S^λ is simple. $\chi^\lambda =$ the character of S^λ

- char$(K) = p > 0$:

If μ is p-regular, S^μ has a unique simple quotient D^μ.

$\beta^\mu =$ the Brauer character of D^μ.

[$\beta^\mu(g) = \mathrm{tr}_{D^\mu}(g)$ if $g \in \mathcal{S}_n$ is p-regular].

μ is p-regular if it does not have p equal parts: $6551 \vdash 17$ is 3-regular, but is not 2-regular.

g is p-regular if p does not divide any cycle length of g.

Decomposition numbers

$d_{\mu,\lambda} := [S^\mu : D^\lambda] \quad = \#D^\lambda$ in a composition series of S^μ,

Decomposition number.

On p-regular elements of $\mathcal{S}_n$,

$$\chi^\lambda = \sum_\mu d_{\lambda,\mu} \beta^\mu$$

EX $\quad D^{(n)} =$ trivial module.

$$[S^{(n-1,1)} : D^{(n)}] = \begin{cases} 1 & p|n \\ 0 & \text{else} \end{cases}$$

If $p|n$ then $\chi^{(n-1,1)} = \beta^{(n-1,1)} + \beta^{(n)}$.

Decomposition matrix

The decomposition matrix $D = [d_{\mu,\lambda}]_{\lambda \vdash n, \mu \vdash_p n}$.

- $d_{\lambda,\mu} \neq 0 \Rightarrow \lambda \geq \mu$
- $d_{\mu,\mu} = 1$.

D is upper uni-triangular.

Some examples See Pictures.

Problem Find decomposition numbers!!

Column removal

Example $p = 2$

$$\ldots = (S^{(5,3)} : D^{(6,2)}) = (S^{(4,2)} : D^{(5,1)}) = (S^{(3,1)} : D^{(4,0)}) = 1$$

General Assume $\hat{\lambda}$ $[\hat{\mu}]$ is obtained from λ $[\mu]$ by removing the first column.

Theorem [G.D.James] If λ, μ have n non-zero parts and $|\lambda| = |\mu|$ then

$$(S^{\lambda} : D^{\mu}) = (S^{\hat{\lambda}} : D^{\hat{\mu}}).$$

Similarly 'row removal' & removal of 'blocks', [S. Donkin]).

Proof (Column removal) The same holds for GL_n. Prove this, then apply Schur functor.

GL_n: Write $\lambda = \lambda_n(1^n) + \widehat{\lambda}$, factorize the Schur polynomial:

$$s_\lambda = (s_{(1^n)})^{\lambda_n} \cdot s_{\widehat{\lambda}}.$$

Similarly for the formal characters of simple modules. Cancel the determinant part.

Two-part partitions

- λ with r parts and $d_{\lambda,\mu} \neq 0 \;\Rightarrow\; \mu$ has $\leq r$ parts.

Theorem [G.D. James '76] $r = 2$:

$$(S^{(n-k,k)} : D^{(n-j,j)}) = \begin{cases} 1 & \binom{n-2j+1}{k-j} \equiv 1 (\text{mod } p) \\ 0 & \text{else} \end{cases}$$

[Column removal]: Get two quarter-infinite matrices which contain the decomposition matrices for all 2-part partitions.

Example $p = 2$ and n even. See pcitures file.

$r \geq 3$ open.

Blocks

If λ, μ are in different blocks, then $d_{\lambda,\mu} = 0$.

Nakayama conjecture

λ and γ are in the same p-block

$\Leftrightarrow \quad \lambda, \gamma$ have the same p-core and the same p-weight;

$\Leftrightarrow \quad \lambda, \gamma$ are in the same 'block' of the decomposition matrix.

Display partitions in B on an abacus with p runners, with $\geq pw$ beads. See the Pictures file.

Equivalences

Suppose $B = B_{\kappa,w}$ is obtained from $\bar{B} = B_{\rho,w}$ by swapping runners $i, i+1$.

- Assume # beads on runners i, $i+1$ differ by $\geq w$.

Theorem [J. Scopes] Swapping runners induces
(i) a bijection on partitions,
(ii) preserves $p-$ regularity and decomposition numbers.
The block algebras B and $\bar{B}$ are Morita equivalent.
For a fixed w, only finitely many blocks (up to Morita equivalence) as n varies.

The first example in the pictures file satisfies the assumption. The second example does not.

The decomposition map

Let $R^n := \sum_{\lambda \vdash n} \mathbb{Z}\chi^\lambda$, $\quad R^n_{br} := \sum_{\mu \vdash_p n} \mathbb{Z}\beta^\mu$. Decomposition map:

$$\xi : R^n \to R^n_{br}, \quad \text{restrict to p-regular elements}$$

Recall On p-regular elements, $\chi^\lambda = \sum_\mu d_{\lambda,\mu}\beta^\mu$.

Decomposition numbers: express the kernel of ξ w.r.to bases χ^λ and β^μ.

Question Other descriptions of ker(ξ)?

$\Lambda = \oplus_{n \geq 0} \Lambda^n$ symmetric functions, characteristic isomorphism

$$\text{char} : \quad \Lambda \quad \rightarrow \quad R := \oplus_{n \geq 0} R^n$$

$$\text{char}(s_\lambda) = \chi^\lambda$$

- $M = GL_n$-module, $M^F =$ its Frobenius twist

$\Rightarrow$ char(χ_{M^F}) is in ker(ξ).

DEF: $\psi^p : \Lambda \rightarrow \Lambda, \quad x_i \rightarrow x_i^p$, ring homomorphism. Then $\psi^p(\chi_M) = \chi_{M^F}$

Via char : $\Lambda \xrightarrow{\sim} R$, get ring homomorphism $\psi^p : R \to R$.

Theorem R^n has $\mathbb{Z}$-basis

$$\{\psi^p(\chi^\lambda) \cdot \chi^\mu : \mu \text{ p-regular}\}$$

The subset of those with $\lambda \neq \emptyset$ are a $\mathbb{Z}$ basis for ker(ξ).

Proof via symmetric functions. If χ^γ occurs in $\psi^p(\chi^\lambda) \cdot \chi^\mu$ then $\gamma \geq \lambda^p \cup \mu$. And $\chi^{\lambda^p \cup \mu}$ occurs with multiplicity ± 1]

δ p-singular $\Rightarrow$ $\delta = \lambda^p \cup \mu$. $\delta \quad \leftrightarrow \quad$ row $[d_{\delta,*}]$ of D

EX $p = 2,\ n = 4.$

$$\psi^2(\chi^{(2)}) = \chi^{(4)} - \chi^{(3,1)} + \chi^{(2,2)}$$

$$\psi^2(\chi^{(1)}) \cdot \chi^{(2)} = \chi^{(4)} + \chi^{(2,2)} - \chi^{(2,1^2)}$$

$$\psi^2(\chi^{(1^2)}) = \chi^{(2,2)} - \chi^{(2,1^2)} + \chi^{(1^4)}$$

Question at the beginning:

$M^{(n-k,k)}$ has Specht filtration with Specht quotients

$$S^{(n)},\ S^{(n-1,1)},\ S^{(n-2,2)},\ \ldots,\ S^{(n-k,k)}.$$

Add corresponding rows of the decomposition matrix. [Depends on 2-adic expansion of n]

$M^{(n-5,3,2)}$ has Specht filtration, quotients from LR rule. Decomposition numbers not known.