Ex Snake Lemma

$M \xrightarrow{f} N$ produce a "Snake diagram"

$$\begin{tikzcd}
 & A' = \ker f \arrow[r] \arrow[d, "\alpha"', "\text{ind}"'] & A = M \arrow[r, "f"] \arrow[d, "\beta", "= id"'] & \mathrm{Im} f = A'' \arrow[r] \arrow[d, "\gamma = 0"] & 0 \\
0 \arrow[r] & C' = M \arrow[r, "1"] & M = C \arrow[r] & 0 = C'' &
\end{tikzcd}$$

$$0 = \ker\beta \longrightarrow \underbrace{\ker\gamma}_{= \mathrm{Im} f} \xrightarrow[\sim]{\partial} \underbrace{\operatorname{coker}\alpha}_{= M/\ker f} \longrightarrow \underbrace{\operatorname{coker}\beta}_{= 0}$$

Recover 1st iso theorem!

chain complex:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

$d_n \circ d_{n+1} = 0 \quad \forall n$.

(could say "d has degree -1")

Cochain complex

$$\cdots \longrightarrow C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} C^{n+2} \longrightarrow \cdots$$

$\delta^{n+1}\delta^n = 0 \quad \forall n$

Rk: Chain complex $\rightsquigarrow$ cochain complex

given the A_n, set $C^n := A_{-n}$, $\delta^n := d_{-n-1}$

or vice versa

Ex: $0 \to A \to B \to C \to 0$

view as chain complex, extend by zeros

Take $A = A_0$, $B = A_{-1}$, $C = A_{-2}$

or variations.

[F additive: induces group hom on Hom-groups].

$$A_{n+2} \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n}$$

$$\longrightarrow F(A_{n+1}) \xrightarrow{F(d_{n+1})} F(A_n) \longrightarrow \cdots$$

Want this to be chain complex:

need functor $\quad F(d_n) \circ F(d_{n+1}) \overset{!}{=} 0$

$F(0) \overset{!}{=} F(d_n d_{n+1})$

Want $F(0) = 0$.

$$\begin{tikzcd}
\arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\equiv"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} & A \arrow[d, "f"] \\
\arrow[r] & B_{n+1} \arrow[r, "d'_{n+1}"] & B_n \arrow[r, "d'_n"] & & B
\end{tikzcd}$$

Req all squares commute.

$\mathcal{C}(R)$ preadditive: Hom sets are abel. groups distributivity.

$\operatorname{im} d_{n+1} \subseteq \ker d_n \quad \forall n$

$$H_n(A) := \ker d_n / \operatorname{Im} d_{n+1} \xrightarrow{H_n(f) = f_*} \frac{\ker d'_n}{\operatorname{Im} d'_{n+1}}$$

$$f_*(a + \operatorname{Im} d_{n+1}) := f_n(a) + \operatorname{Im} d'_{n+1}$$

- Is $f_n(a) \in \ker d'_n$?

$a \in \ker d_n$

$$d'_n f_n(a) = f_{n-1} d_n(a) = 0$$

- Is f_* well defined?

Let $a_1 + \operatorname{Im} d_{n+1} = a_2 + \operatorname{Im} d_{n+1}$

$$a_1 - a_2 = d_{n+1}(x)$$

$$f_n(a_1) - f_n(a_2) = f_n(a_1 - a_2) = f_n d_{n+1}(x) = d'_{n+1} f_{n+1}(x) \in \operatorname{Im} d'_{n+1}$$

So $f_n(a_1) + \operatorname{Im} d'_{n+1} = f_n(a_2) + \operatorname{Im} d'_{n+1}$

✓

What is homology? A lin. transf.

$$\cdots \; 0 \to 0 \to \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \to 0 \to \cdots$$

$$0 \qquad 0 \qquad \operatorname{Ker} A \qquad \mathbb{R}^m / \operatorname{Im} A$$

Have defined H_n on $\mathbb{A}$ chain cx

on f chain map

Functor: to check ✓ $H_n(f\circ g) = H_n(f)\circ H_n(g)$

$H_n(1) = 1$

! H_n is additive: $H_n(f \pm g) = H_n(f) \pm H_n(g)$

$f : \mathbb{A} \longrightarrow \mathbb{A}'$ chain map

ker f :

$$\begin{tikzcd}
\ker f_{n+1} \arrow[r, "d"] \arrow[d] & \ker f_n \arrow[d, "\mathrm{incl}"] & \\
A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \\
 \arrow[r, "d'_{n+1}"'] & &
\end{tikzcd}$$

$d_{n+1}(\ker f_{n+1}) \subseteq \ker f_n$

$f_n d_{n+1}(x)$

$= d'_{n+1} \underbrace{f_{n+1}(x)}_{=0}$

5.3

$$\begin{tikzcd} A_n/\ker f_n \arrow[r, "\bar{d}_n"] \arrow[d, "\rho_n"'] & A_{n-1}/\ker f_{n-1} \arrow[d, "\rho_{n-1}"] \\ \operatorname{Im} f_n \arrow[r, "d_n'"] & \operatorname{Im} f_{n-1} \end{tikzcd}$$

$\operatorname{Im} f_n \subset A_n'$

$\rho_n(x+\ker f_n)$
$= f_n(x)$
$x \in A_n$

ρ_n is iso.

(ρ_n) is a chain map:

$d_n'\rho_n(x+\ker f_n)$
$= d_n' f_n(x) = f_{n-1} d_n(x)$

$\rho_{n-1}(\bar{d}_n(x+\ker f_n)) = \rho_{n-1}(d_n(x)+\ker f_n)$
$= f_{n-1}(d_n(x))$

5.4

by exercise from exam.

$$\begin{tikzcd}
 & & \arrow[d, "d_{n+1}"] & \arrow[d] & \\
0 \arrow[r] & A'_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & A_n \arrow[r, "p_n"] \arrow[d, "d_n"] & A''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & A'_{n-1} \arrow[r, "i"] \arrow[d, "d'_{n-1}"] & A_{n-1} \arrow[r, "p"] & A''_{n-1} \arrow[r] & 0 \\
 & \; & & &
\end{tikzcd}$$

$H_n A'' = \dfrac{\operatorname{Ker} d''_n}{\operatorname{Im} d''_{n+1}}$

$H_{n-1}(A') = \dfrac{\operatorname{Ker} d'_{n-1}}{\operatorname{Im} d'_n}$

Snake Lemma gives

$\tilde{\partial} : \operatorname{Ker} d''_n \longrightarrow A'_{n-1} / \operatorname{Im} d'_n$

This induces

$\partial : \operatorname{Ker} d''_n / \operatorname{Im} d''_{n+1} \longrightarrow A'_{n-1} / \operatorname{Im} d'_n$

check well defined $\quad x + \operatorname{Im} d''_{n+1} \to \tilde{\partial}(x)$

To show: it maps into $\dfrac{\operatorname{Ker} d'_{n-1}}{\operatorname{Im} d'_n}$