

Ex Snake Lemma

$M \xrightarrow{f} N$  podna a "Snake diagram"

$$\begin{array}{ccccccc}
 A' = \ker f & \longrightarrow & A = M & \xrightarrow{f} & \text{Im } f = A'' & \longrightarrow & 0 \\
 \text{ind } \alpha \downarrow & & \beta \downarrow = \text{id} & & \downarrow \gamma = 0 & & \\
 0 \longrightarrow & C' = M & \xrightarrow{\tau} & M & \longrightarrow & 0 & \\
 & & & \parallel & & \parallel & \\
 & & & C & & C'' & \\
 0 = \ker \beta & \longrightarrow & \ker \gamma & \xrightarrow{\cong} & \text{coker } \alpha & \longrightarrow & \text{coker } \beta \\
 & & \downarrow & & \parallel & & \parallel \\
 & & \text{Im } f & & M / \ker f & & 0
 \end{array}$$

Recuer 1st iso theorem!

chain complex:

$$\dots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \dots$$

$$d_n \circ d_{n+1} = 0 \quad \forall n.$$

(could say "d has degree -1")

Cochain complex

$$\dots \longrightarrow C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} C^{n+2} \longrightarrow \dots$$

$$\delta^{n+1} \circ \delta^n = 0 \quad \forall n$$

Rk: Chain complex  $\rightsquigarrow$  cochain complex

given the  $A_n$  | set  $C^n := A_{-n}$   $\delta^n := d_{-n-1}$   
or vice versa

Ex:  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$

view as chain complex, extend by zeros

Take  $A = A_0$  |  $B = A_{-1}$  |  $C = A_{-2}$   
or variation.

[  $\mathbb{F}$  additive, induced group hom on Hom - groups ]

$$A_{n+2} \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n}$$

$$\rightarrow \mathbb{F}(A_{n+1}) \xrightarrow{\mathbb{F}(d_{n+1})} \mathbb{F}(A_n) \rightarrow \dots$$

want this to be chain complex:

need  $\mathbb{F}(d_n) \circ \mathbb{F}(d_{n+1}) \stackrel{!}{=} 0$

$\mathbb{F}(0) = \mathbb{F}(d_n d_{n+1})$   
 want  $\mathbb{F}(0) = 0$ .

$$\begin{array}{ccccccc}
 \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & A \\
 \downarrow & & & \downarrow f_{n+1} \cong \downarrow f_n & & & \downarrow f & \text{Req} \\
 \longrightarrow & B_{n+1} & \xrightarrow{d'_{n+1}} & B_n & \xrightarrow{f'_n} & & B & \text{all} \\
 & & & & & & & \text{squares} \\
 & & & & & & & \text{commute.}
 \end{array}$$

$\mathcal{L}(R)$  preconditions: How rich are abel. groups distributivity.

$$\overline{\text{im } d_{n+1} \subseteq \ker d_n} \quad \forall n$$

$$H_n(A) := \ker d_n / \text{im } d_{n+1} \xrightarrow{H_n(f) = f_x} \frac{\ker d'_n}{\text{im } d'_{n+1}}$$

$$f_x(a + \text{im } d_{n+1}) := f_n(a) + \text{im } d'_{n+1}$$

• Is  $f_n(a) \in \ker d'_{n+1}$ ?

$$d'_n f_n(a) = f_{n-1, n}(a) = 0 \quad a \in \ker d_n$$

• Is  $f_x$  well defined?

$$\text{Let } a_1 + \text{im } d_{n+1} = a_2 + \text{im } d_{n+1}$$

$$a_1 - a_2 = d_{n+1}(x)$$

$$f_n(a_1) - f_n(a_2) \in \text{im } f_n(d_{n+1}(x)) = \text{im } d'_{n+1}(f_n(x)) \in \text{im } d'_{n+1}$$

$$\text{So } f_n(a_1) + \text{im } d'_{n+1} = f_n(a_2) + \text{im } d'_{n+1}$$

✓

What is homology?

A lin. trans.

$$\begin{array}{ccccccc}
 \dots & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m & \longrightarrow & 0 & \longrightarrow & \dots \\
 & 0 & & 0 & & \text{Ker } A & & \mathbb{R}^m & & & & \\
 & & & & & & & \hline & & & & & \text{Im } A & & & & & & 
 \end{array}$$

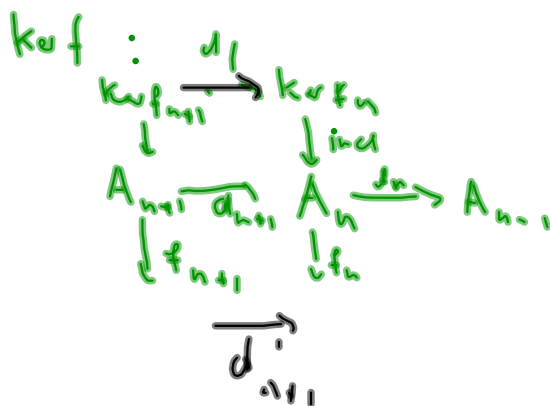
Have defined  $H_n$  on  $A$  chain  $\mathcal{C}$   
 on  $f$  chain map

Functor: to check  $H_n(f \circ g) = H_n(f) \circ H_n(g)$

$$H_n(1) = 1$$

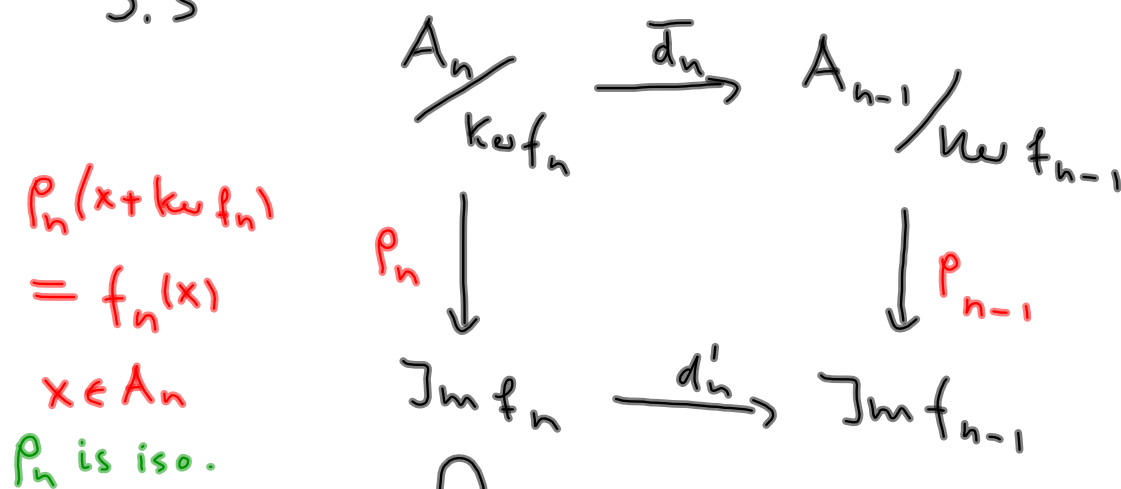
!  $H_n$  is additive:  $H_n(f \pm g) = H_n(f) \pm H_n(g)$

$f: A \rightarrow A'$  chain map



$$\begin{aligned}
 & d_{n+1}(\text{ker } f_{n+1}) \\
 & \subseteq \text{ker } f_n \\
 & f_n d_{n+1}(x) \\
 & = d_{n+1}' \underbrace{f_{n+1}(x)}_{=0} \\
 & = 0
 \end{aligned}$$

5.3

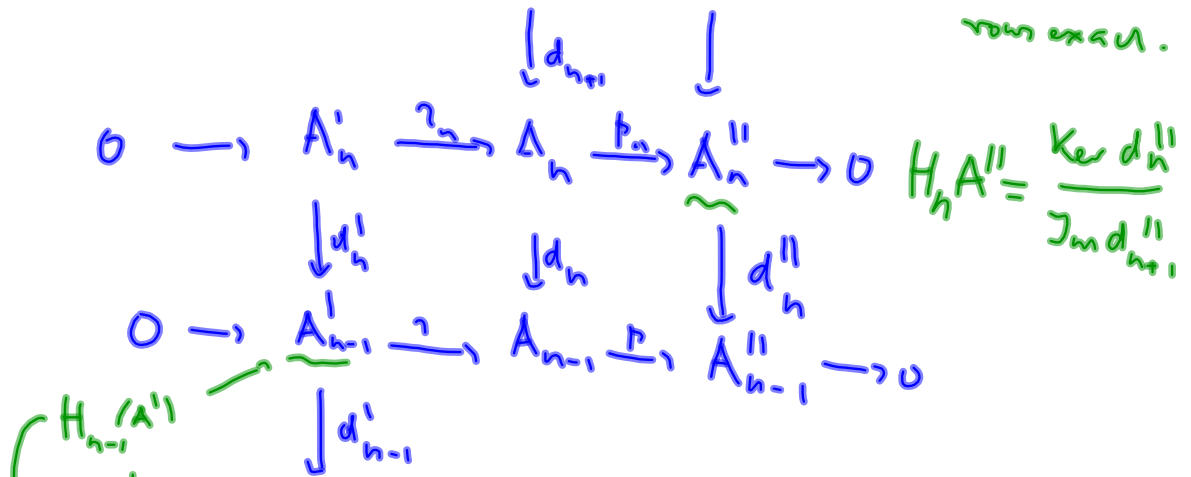


$(\rho_n)$  is a chain map:  $d'_n \rho_n(x + \ker f_n) = d'_n f_n(x) = f_{n-1} d_n(x)$

$$\begin{aligned}
 \rho_{n-1}(\bar{d}_n(x + \ker f_n)) &= \rho_{n-1}(d_n(x) + \ker f_{n-1}) \\
 &= f_{n-1}(d_n(x))
 \end{aligned}$$

5.4

by exercise  
row exact.



Snake Lemma gives

$$\tilde{\partial} : \text{Ker } d_n'' \longrightarrow A_{n-1}' / \text{Im } d_n'$$

This induces

$$\partial : \text{Ker } d_n'' / \text{Im } d_{n+1}'' \longrightarrow A_{n-1}' / \text{Im } d_n'$$

check  
well defined

$$x + \text{Im } d_{n+1}'' \longrightarrow \tilde{\partial}(x)$$

To show: it maps into  $\frac{\text{Ker } d_{n-1}'}{\text{Im } d_n'}$