$$A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} \cdots \qquad d_n d_{n+1} = 0$$

$$H_n(A) = \ker d_n / \operatorname{Im} d_{n+1}$$

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0 \quad \text{(complexes)}$$

$$\longrightarrow H_n(A') \xrightarrow{i_*} H_n(A) \xrightarrow{p_*} H_n(A'') \xrightarrow{\partial} H_{n-1}(A') \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(A'') \xrightarrow{\partial} \cdots$$

∂ defined in 5.4.

Check exact.

Eg: $\operatorname{Im} i_* = \ker p_*$: "$\subseteq$" $p_* \circ i_* = (pi)_* = 0$

"$\supseteq$"

$$\begin{tikzcd}
 & \arrow[d] & a\ A_{n+1} \arrow[r, "p_{n+1}"] \arrow[d] & A''_{n+1}\ a'' \arrow[d, "d''_{n+1}"] & \\
0 \arrow[r] & A'_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & \boxed{A_n}\ z \arrow[r, "p_n"] \arrow[d, "d_n"] & A''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
 & a' & & &
\end{tikzcd}$$

$H_n(A) = \dfrac{\ker d_n}{\operatorname{Im} d_{n+1}}$

Let $p_*(z + \operatorname{Im} d_{n+1}) = 0$, $z \in \ker d_n$

Want $z + \operatorname{Im} d_{n+1} = i(a') + \operatorname{Im} d_{n+1}$, $a' \in \ker d'_n$

$\Downarrow$

$\underline{p_n(z)} = d''_{n+1}(a'')$

$= d''_{n+1} p_{n+1}(a) = \underline{p_n d_{n+1}(a)}$

$p_n(z - d_{n+1}(a)) = 0$ $\quad \ker p_n = \operatorname{Im} i_n$

$z - d_{n+1}(a) = i_n(a')$ $\quad a' \in A'_n$

need $a' \in \ker d_n$

$i_{n-1} d'_n(a') = d_n i_n(a') = d_n(z - d_{n+1}(a))$

$= d_n(z) - 0$

$z \in \ker d_n$ $\quad$ "0"

So $d'_n(a') = 0$ as well

i is 1-1.

Homotopy

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, dashed, "s_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"'] \\ \cdots \arrow[r, "d'_{n+1}"] & A_n \arrow[r, "d'_n"] & A'_{n-1} \end{tikzcd}$$

squares commute

$f \sim 0 \iff \exists\, s_n$ with

$$f_n = s_{n-1} d_n + d_{n+1} s_n \quad \forall n$$

$f \sim g \iff f - g \sim 0$

$\sim$ is an eq. relation.

Ex

$$\begin{tikzcd}
& 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & A_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & A_1 \arrow[r] \arrow[d, "0"] & 0 \cdots \\
\cdots 0 \arrow[r] & A'_{-1} \arrow[r, "d'_{-1}"] & A'_0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

= a chain map

$R = k[x]/(x^2)$ $\quad \bar{x} = x + (x^2)$

$$\begin{tikzcd}
& \cdots 0 \arrow[r] \arrow[d, "0"] & R \arrow[r, "d_0"] \arrow[d, "f_0"] \arrow[dl, dotted, "s_0"] & R \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "s_1"] & 0 \\
0 \arrow[r] & R \arrow[r, "d'_{-1}"] & R \arrow[r] & 0 & \cdots
\end{tikzcd}$$

$1 \longmapsto \bar{x}$ $\quad 1 \longmapsto \bar{x}$

Let $f_0 = id$

Is $f \sim 0$? No.

$\underset{\text{does not}}{1} = \underbrace{s_0 d'_{-1} + s_1 d_0}_{\text{map into } R\bar{x}}$

if $f_0 = (1 \to \bar{x})$, $f \sim 0$

$s_0 = id$, $s_1 = 0$

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

$$\cdots * \longrightarrow * \longrightarrow B \otimes_R A_1 \xrightarrow[\text{usually not 1-1}]{1} B \otimes_R A_2 \longrightarrow B \otimes_R A_3 \longrightarrow 0$$

X_n projective

$$\begin{tikzcd} X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "d_1"] & X_0 \arrow[r, "\varepsilon"] \arrow[d, dotted, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ X_2' \arrow[r] & X_1' \arrow[r] & X_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] & 0 \end{tikzcd}$$

exact ...

Claim: $\exists$ chain map $\bar{f}: X_A \longrightarrow X_{A'}$ lifting f. Any other $\bar{g}$ lifting f is $\bar{f} \sim \bar{g}$.

Proof: X_0 is projective, ε' is onto

$\exists f_0$ with $\varepsilon' f_0 = f\varepsilon$.

Inductively: $f_0, f_1, \dots, f_n$ found, $d_i' f_i = f_{i-1} d_i$

$$\begin{tikzcd} \arrow[r, "d_{n+1}"] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\ \arrow[r] & X_n' \arrow[r, "d_n'"] & \cdot \end{tikzcd}$$

$d_n' f_n d_{n+1}$
$= f_{n-1} d_n d_{n+1}$
$= 0$

$\operatorname{Im}(f_n d_{n+1}) \subseteq \ker d_n'$
$= \operatorname{im} d_{n+1}'$

$$\begin{tikzcd} & X_{n+1} \arrow[dl, dotted, "\exists f_{n+1}"'] \arrow[d, "f_n d_{n+1}"] & \\ X_{n+1}' \arrow[r, "d_{n+1}'"] & \operatorname{Im} d_{n+1}' \arrow[r] & 0 \end{tikzcd}$$

To show: If also (g_n) lifts f
$(f_n) \sim (g_n)$.

$$\begin{tikzcd}
{} \arrow[r] & X_n \arrow[r, "d_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & {} & X_0 \arrow[r, "\varepsilon"] \arrow[dl, "s_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & A \arrow[d, "f"] \arrow[dl, dotted, "0"'] \\
{} \arrow[r] & X_n' \arrow[r] & {} \arrow[r, "d_1'"] & X_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] & 0
\end{tikzcd}$$

$\urcorner$ $f\varepsilon = \varepsilon' f_0 = \varepsilon' g_0 \;\therefore\; \varepsilon'(f_0 - g_0) = 0$

$\mathrm{Im}(f_0 - g_0) \subseteq \mathrm{Ker}\,\varepsilon' = \mathrm{im}\, d_1'$

X_0 is projective: $\exists\, s_0 : X_0 \longrightarrow X_1'$ with

$d_1' s_0 = f_0 - g_0$, take $s_{-1} = 0$

$\Longrightarrow f_0 - g_0 = d_1' s_0 + s_{-1}\varepsilon$ ✓

Repeat (inductively)

$$\longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

exact, P_i projective

complex P_A $\quad \cdots P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to 0$

homology A at $n=0$

$n \neq 0$: exact.

$T: R\text{-Mod} \longrightarrow Ab$ additive functor

Left derived functor:

$TP_A \qquad TP_n \xrightarrow{Td_n} TP_{n-1} \to \cdots$

is a complex: $\quad Td_n \circ Td_{n+1} = T(d\,d_{n+1})$

Def $\quad L_nT: R\text{-Mod} \to Ab$ $\quad = T(0) = 0$

$A \longrightarrow$ n-th homology of TP_A

$\operatorname{Ker} Td_n / \operatorname{Im} Td_{n+1}$.

On maps:

$f: A \to B$. Lift to chain map $P_A \xrightarrow{\bar f} P_B$

(use 6.11) Apply T, get chain map

$$T\bar f: TP_A \to TP_B$$

DEF $(L_nT)f: H_n(TP_A) \to H_n(TP_B)$

map induced by $T\bar f$

$\bar f = (f_n) \quad T\bar f = (Tf_n)$

$(L_nT)f = (Tf_n)_*$

well defined:

If $(f_n), (f_n')$ lift f, need

$$(Tf_n)_* = (Tf_n')_*.$$