$A \in R\text{-Mod}$

$\cdots \to P_1 \to P_0 \to A \to 0$

proj resolution. $T : R\text{-Mod} \to Ab$ addition

$\xrightarrow{Td_{n+1}} TP_n \xrightarrow{Td_n} TP_{n-1} \to \cdots \to TP_0 \to 0$

$L_nT(A) = \ker Td_n / \operatorname{Im} Td_{n+1}$

Ex:

$T = B \otimes_R (-)$. Eg $B = \mathbb{Z}/2\mathbb{Z}$

$R = \mathbb{Z}$ $A = \mathbb{Z}/\mathbb{Z}m$ $(0 \neq m)$

Projective solution:

$$\cdots 0 \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} = P_0 \to A \to 0$$

$0 = P_2$, $\mathbb{Z} = P_1$, $d_1(z) = zm$ 1-1

$$B \otimes 0 \xrightarrow{0} B \otimes \mathbb{Z} \xrightarrow{1 \otimes d_1} B \otimes \mathbb{Z} \to 0$$

TP_A, $B \otimes \mathbb{Z} = B \otimes P_1$

$(L_1T)A = \ker(1 \otimes d_1) / \operatorname{im} = \ker(1 \otimes d_1)$

$(1 \otimes d_1)(\bar{z} \otimes r) = \bar{z} \otimes mr$, $\bar{z} \in B$, $r \in \mathbb{Z}$, $mr \in \mathbb{Z}$

$= \bar{z}m \otimes r$

$$= \begin{cases} 0 & m \text{ even} \\ \bar{z} \otimes r & m \text{ odd} \end{cases}$$

So $(L_1T)A = \begin{cases} B \otimes \mathbb{Z} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$

Case $T = B \otimes_R (-)$

$$L_n T = \mathrm{Tor}_n^R(B, -)$$

$L_n T(A)$
$= \mathrm{Tor}_n^R(B,-)(A)$
$= \mathrm{Tor}_n^R(B, A)$

Case $T = (-) \otimes_R A$

$$L_n T = \mathrm{Tor}_n^R(-, A)$$

$L_n T(B) = \mathrm{Tor}_n^R(B, A)$

? are these the same?

6.3

$$\begin{tikzcd}
\arrow[r] & P_n \arrow[r] \arrow[d, "\eta_n"] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\eta_0"] & A \arrow[r] \arrow[d, "id"] & 0 \\
\arrow[r] & P'_n \arrow[r] & \cdots \arrow[r] & {} \arrow[r] & P'_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

two proj resolutions. Get L_nT, L'_nT

Lift 1_A to chain map (see 6.1)

$\Rightarrow$ $(T\eta_n)$ is chainmap, lifts $T(id) = id_{TA}$

induces a map $(L_nT)A \xrightarrow[\tau_A]{(T\eta_n)_*} (L'_nT)A$

To show τ_A is iso.

And τ is a natural transformation.

6.4

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[dr, "d_2"'] & P_1 \arrow[r, "d_1"] \arrow[dr, two heads] & P_0 \arrow[r, "d_0"] & A \\
 & & \ker d_1 \arrow[u] & \ker d_0 = K_0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\Delta_2"'] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r] & K_0 \arrow[r] & 0
\end{tikzcd}$$
proj res.

$P_2 = Q_1$, $d_2 = \Delta_1$, $P_1 = Q_0$

$$(L_n T)(K_0) = \ker T\Delta_n \big/ \operatorname{Im} T\Delta_{n+1} = \frac{\ker Td_{n+1}}{\operatorname{Im} Td_{n+2}} = (L_{n+1}T)A$$

$$0 \to P' \to P \underset{g}{\overset{f}{\rightleftarrows}} P'' \to 0 \qquad f \circ g = 1_{P''}$$

$$TP' \to TP \underset{Tg}{\overset{Tf}{\rightleftarrows}} TP'' \qquad T(f)T(g) = T(fg) = T(1) = 1_{TP''}$$

What does 6.5 say?

Eg $T = B \otimes_R (-)$.

will see $L_0T \simeq T$, get long exact sequence

$$\cdots \to \mathrm{Tor}_2^R(B, A'') \to \mathrm{Tor}_1^R(B, A') \to \mathrm{Tor}_1^R(B, A) \to \mathrm{Tor}_1^R(B, A'') \to B \otimes A' \to B \otimes A \to B \otimes A'' \to 0$$

V. important. In practice, might look for terms which are 0

Horseshoe Lemma: Given SES and proj resolutions for end terms $\rightsquigarrow$ proj resolution for the middle.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P'_1 \arrow[d] & & P''_1 \arrow[d] & \\
 & P'_0 \arrow[d] & & P''_0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}
$$

Basic step:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K' \arrow[d] \arrow[r, dashed] & K \arrow[r, dashed] & K'' \arrow[d] \arrow[r] & 0 \\
0 \arrow[r] & P' \arrow[r, "j"] \arrow[d, "d'"] & P \arrow[r, "q"] \arrow[d, "d"] & P'' \arrow[r] \arrow[d, "d''"] \arrow[dl, dotted, "\sigma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] & A \arrow[r, "p"] & A'' \arrow[r] & 0
\end{tikzcd}
$$

P'' projective

$\exists \sigma$

$p \circ \sigma = d''$

$P = P' \oplus P''$

$$d(a', a'') := i d'(a') + \sigma(a'')$$

$\square$?

$$p d(a', a'') = \underbrace{p i d'(a')}_{0} + p\sigma(a'')$$

$$q(a', a'') = a''$$

$$d'' q(a', a'') = d''(a'')$$

Snake Lemma

$0 \to K' \to \ker d \to K'' \to \operatorname{Coker} d' \to \operatorname{Coker} d \to \operatorname{Coker} d'' \to 0$

$\operatorname{Coker} d' = 0$, $\operatorname{Coker} d'' = 0$

$\Rightarrow d$ is onto

$$\begin{tikzcd}
 & P_2' \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d] & P_2'' \arrow[d, dotted] & \\
 & P_1' \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

$\therefore$ SES of complexes

$$0 \longrightarrow P_{A'} \longrightarrow P_A \longrightarrow P_{A''} \longrightarrow 0$$

P_n' projective $\Rightarrow$ all rows in $(*)$ split

Apply T. By Exercise, get split exact seq

$$0 \longrightarrow TP_{A'} \longrightarrow TP_A \longrightarrow TP_{A''} \longrightarrow 0$$

§5: exact seq of complexes $\rightsquigarrow$ long exact sequence of homology.

SYZYGY

Rotman's labelling
differs from
other authors

Right derived functor:

Main interest: $T = \mathrm{Hom}_R(B, -)$

left exact, usually not exact.

$$0 \to A' \to A \to A'' \to 0$$

gives $0 \to \mathrm{Hom}(B, A') \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, A'')$

$\to R^1\mathrm{Hom}(B, A') \to \dots$

$\quad\;\; \| $

$\mathrm{Ext}^1(B, A') \to \dots$

$R^n\mathrm{Hom}(B, -)$ will be $\mathrm{Ext}^n(B, -)$.