

$$A \in R\text{-Mod}$$

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

proj resolution. $T: R\text{-Mod} \rightarrow \text{Ab}$
addition

$$\xrightarrow{Td_n} TP_n \xrightarrow{Td_{n-1}} TP_{n-1} \rightarrow \dots \rightarrow TP_0 \rightarrow 0$$

$$L_n T(A) = \ker Td_n / \text{Im } Td_{n+1}$$

Ex:

$$T = B \otimes_R (-). \quad \text{Ex } B = \mathbb{Z}/2\mathbb{Z}$$

$$R = \mathbb{Z} \quad A = \mathbb{Z}/\mathbb{Z}m \quad (0 \neq m)$$

Projective solution:

$$\begin{array}{ccccccc} \dots & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \xrightarrow{d_1} & \mathbb{Z} = P_0 \rightarrow A \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & P_2 & & P_1 & & \mathbb{Z} \end{array}$$

$$\begin{array}{ccccccc} & & B \otimes 0 & \xrightarrow{0} & B \otimes \mathbb{Z} & \xrightarrow{1 \otimes d_1} & B \otimes \mathbb{Z} \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & TP_A & & B \otimes P_1 & & \end{array}$$

$$(L_n T)A = \ker(1 \otimes d_1) / \text{Im} = \ker(1 \otimes d_1)$$

$$(1 \otimes d_1)(\bar{z} \otimes r) = \bar{z} \otimes mr$$

$$= \begin{cases} 0 & m \text{ even} \\ \bar{z} \otimes r & m \text{ odd} \end{cases}$$

$$\text{So } (L_n T)A = \begin{cases} B \otimes \mathbb{Z} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

Case $T = B \otimes_R (-)$

$$L_n T = \text{Tor}_n^R(B, -)$$

$$\begin{aligned} L_n T(A) &= \text{Tor}_n^R(B, -)(A) \\ &= \text{Tor}_n^R(B, A) \end{aligned}$$

Case $T = (-) \otimes_R A$

$$L_n T = \text{Tor}_n^R(-, A)$$

$$L_n T(B) = \text{Tor}_n^R(B, A)$$

← ? are these the same?

6.3

$$\rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow A \rightarrow 0$$

$$\begin{array}{ccccccc} \downarrow \tau_n & & \cong \tau_0 \downarrow \cong & \cong \downarrow \text{id} & & & \\ \rightarrow P'_n \rightarrow & \dots & \rightarrow & P'_0 \rightarrow & A \rightarrow & 0 & \end{array}$$

two proj resolutions. Get $L_n T, L'_n T$

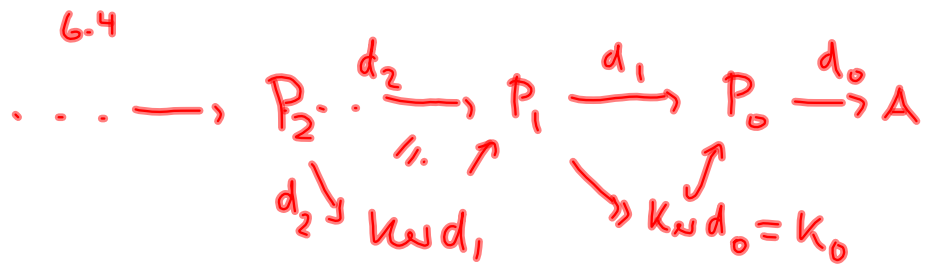
Lift 1_A to chain map (see 6.1)

$\Rightarrow (T\tau_n)$ is chain map, lifts $T(\text{id}) = \text{id}_{TA}$

induces a map $(L_n T)A \xrightarrow{(T\tau_n)_*} (L'_n T)A$

To show τ_A is iso.

And τ is a natural transformation.



$$\dots \rightarrow P_2 \xrightarrow{d_2} P_1 \rightarrow k_0 \rightarrow 0 \text{ proj. res.}$$

$\begin{matrix} \Delta_2 \\ \parallel \\ Q_2 \end{matrix}$

$\begin{matrix} \Delta_1 \\ \parallel \\ Q_1 \end{matrix}$

$\begin{matrix} \parallel \\ Q_0 \end{matrix}$

$$(L_n T) / k_0 = \frac{kw T \Delta_n}{Im T \Delta_{n+1}} = \frac{kw T d_{n+1}}{Im T d_{n+2}} = (L_{n+1} T) A$$

$$0 \rightarrow P' \rightarrow P \xrightarrow{f} P'' \rightarrow 0 \quad f \circ g = 1_{P''}$$

$\leftarrow \dots$
 g

$$TP' \rightarrow TP \xrightarrow{Tf} TP''$$

$\leftarrow \dots$
 Tg

$T(f)T(g)$
 $= T(fg) = T(1)$
 $= 1_{TP''}$

What does 6.5 say?

Ex $T = B \otimes_R (-)$.

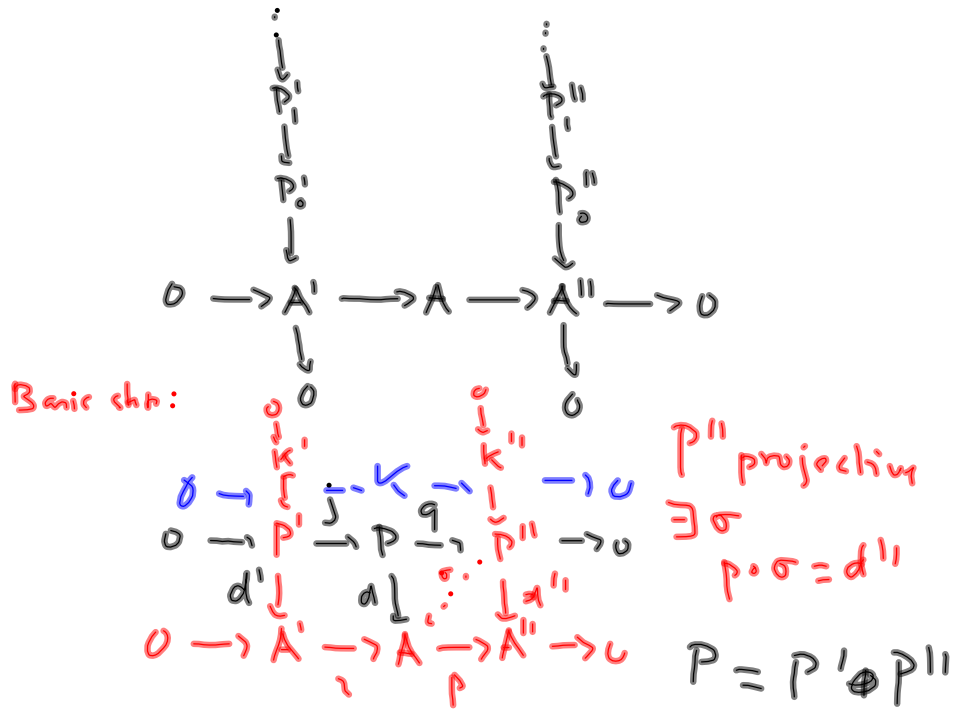
will see $L_0 T \cong T$, get long exact sequence

$$\hookrightarrow B \otimes A' \rightarrow B \otimes A \rightarrow B \otimes A'' \rightarrow 0$$

$$\dots \rightarrow \text{Tor}_2^R(B, A'') \rightarrow \text{Tor}_1^R(B, A') \rightarrow \text{Tor}_1^R(B, A) \rightarrow \text{Tor}_0^R(B, A'') \rightarrow$$

V. important. In practice, might look for terms which are 0

Horseshoe Lemma: Given SES and proj resolutions for end terms \rightarrow proj resolution for the middle.



$$d(a', a'') := id'(a') + \sigma(a'')$$

$$\square? \quad pd(a', a'') = p(id'(a') + \sigma(a''))$$

$$q(a', a'') = a''$$

$$d''q(a', a'') = d''(a'')$$

Snake Lemma

$$0 \rightarrow K' \xrightarrow{d'} K \xrightarrow{d''} K'' \rightarrow 0$$

$$\rightarrow \text{Coker } d' \rightarrow \text{Coker } d \rightarrow \text{Coker } d'' \rightarrow 0$$

$\Rightarrow d$ is onto

$$\begin{array}{ccccccc}
 P_2' & \rightarrow & P_2 & \rightarrow & P_2'' & & \\
 \downarrow & & \downarrow & & \vdots & & \\
 P_1' & \rightarrow & P_1 & \rightarrow & \vdots & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & P_0' & \rightarrow & P_0 & \rightarrow & P_0'' \rightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & A' & \rightarrow & A & \rightarrow & A'' \rightarrow 0
 \end{array}
 \quad (*)$$

\therefore SES of complexes

$$0 \rightarrow P_{A'} \rightarrow P_A \rightarrow P_{A''} \rightarrow 0$$

P_n' projective \Rightarrow all rows in (*) split

Apply T By Exercise, get split exact seq

$$0 \rightarrow TP_{A'} \rightarrow TP_A \rightarrow TP_{A''} \rightarrow 0$$

$\S 5$: exact seq of complexes \rightsquigarrow long exact sequence of homology.

Syzygy

Rotman's labelling
different from
other authors

Right derived functor:

Main interest: $T = \text{Hom}_R(B, -)$

left exact, usually not exact.

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

$$\text{so } 0 \rightarrow \text{Hom}(B, A') \rightarrow \text{Hom}(B, A) \rightarrow \text{Hom}(B, A'')$$

$$\hookrightarrow R^1 \text{Hom}(B, A') \rightarrow \dots$$

$$\text{Ext}^1(B, A') \rightarrow \dots$$

$R^h \text{Hom}(B, -)$ will be $\text{Ext}^h(B, -)$.