

Problem sheet:

deadline 15 April 2012

[~ 4 sheets in total.
Do $\geq 50\%$ (correct)].

recommen: do problems as we go along.

$$\begin{array}{ccccccc}
 4.6 & 0 & \rightarrow & A & \xrightarrow{\iota} & D & \xrightarrow{p_X} & X & \rightarrow & 0 \\
 & & & \downarrow \text{id} & & \downarrow p_B & & \downarrow \alpha & & \\
 & 0 & \rightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\iota} & C & \rightarrow & 0
 \end{array}$$

$$D = \{ (b, x) \in B \oplus X : \beta(b) = \alpha(x) \}$$

$$p_X(b, x) = x \quad p_B(b, x) = b$$

$\gamma(a) := (a, 0)$ γ is 1-1, p_X is onto

$$\begin{array}{ccc|ccc}
 \text{LHS square:} & a & \rightarrow & (a, 0) & & \text{RHS square} \\
 & \downarrow & & \downarrow & & (b, x) \rightarrow X \\
 & a & \rightarrow & a & & \downarrow \\
 & & & & & b \rightarrow \beta(A) = \alpha(X)
 \end{array}$$

$$\text{Im } \gamma = \ker p_X$$

$$\geq \text{Let } p_X(b, x) = 0, \text{ so } x = 0 \quad \Bigg| \quad \subseteq \text{ easy.}$$

$$\Rightarrow \alpha p_X(b, 0) = \alpha(0) = 0$$

$$\beta p_B(b, 0) = \beta(b) \quad b \in V, \alpha \beta = A$$

$$(b, 0) = \gamma(b) \quad \sim \gamma(b)$$

P projection :
$$\begin{array}{c} \exists \delta \dots \\ \downarrow \alpha \\ B \xrightarrow{\beta} C \rightarrow 0 \end{array}$$

\iff Hom $(P, -)$ is exact.

4.7 : P projective $\iff \forall 0 \rightarrow A \rightarrow B \xrightarrow{\pi} P \rightarrow 0$ is split.

" \implies "
$$0 \rightarrow A \rightarrow B \xrightarrow{\beta} P \rightarrow 0$$
 in δ : Sequence is split $P \circ j = I_P$

" \impliedby " Let $B \xrightarrow{\beta} C \rightarrow 0$ SES $0 \rightarrow A \xrightarrow{\gamma} B \xrightarrow{\beta} C \rightarrow 0$

Apply 4.6
$$\begin{array}{ccccccc} 0 & \rightarrow & A & \xrightarrow{\gamma} & D & \xrightarrow{P} & P \rightarrow 0 \\ & & \text{id} \downarrow & \cong & \downarrow \beta & = & \downarrow \alpha \\ 0 & \rightarrow & A & \rightarrow & B & \xrightarrow{\pi} & C \rightarrow 0 \end{array}$$

By hypothesis, top row is split $P \circ j = I_P$

Try $\delta = P \circ j : P \rightarrow B$

$\pi \circ P \circ j = \alpha \circ P \circ j = \alpha \circ I_P = \alpha$

4.8 a) $P = Q \oplus X$, P projective

$$\begin{array}{ccc}
 & \begin{array}{c} \uparrow \lambda \\ Q \end{array} & \\
 \begin{array}{c} \exists \gamma \\ \downarrow \gamma \end{array} & \begin{array}{c} \downarrow \alpha \\ C \end{array} & \\
 B & \xrightarrow{\beta} & C \rightarrow 0
 \end{array}$$

$\beta \circ \gamma = \alpha$

P projective: $\exists \gamma$ with $\beta \circ \gamma = \alpha$
 $\Rightarrow \beta \circ \gamma \circ \lambda = \alpha \circ \lambda = \alpha|_Q = \alpha$.
 So Q is projective.

(b) \Leftarrow F free \Rightarrow projective 4.3 \Rightarrow any summand is projective by a)
 \Rightarrow Assume P is projective.

4.2: \exists free module F and an epi $F \rightarrow P \rightarrow 0$
 \Rightarrow have $0 \rightarrow A \rightarrow F \rightarrow P \rightarrow 0$ SES
 By 4.7 this is split $\therefore F \cong P \oplus A$.

Ex $R = k\langle g \rangle$ $\langle g \rangle$ group, $g^2 = 1 \neq g$

assume $\text{char } k \neq 2$

$$\left(\frac{1}{2}(1 \pm g)\right)^2 = \frac{1}{4}(1 + g^2 \pm 2g) = \frac{1}{4}(2(1 \pm g))$$

$$1 = e_0 + e_1, \quad e_0 e_1 = 0$$

By Linear Algebra

$$R = R_{e_0} \oplus R_{e_1}, \quad R_{e_i} \text{ is 1-dim}$$

R is free, so R_{e_i} is projective.

not free: R is the smallest $\neq 0$ projective mod

$$\dim_k R = 2, \quad \dim_k R_{e_i} = 1 \quad R_{e_i} \neq R$$

$$R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$$

as R -modules

$\mathbb{Z}/2\mathbb{Z}$ is projective as R -module
not free

4.9 Application:

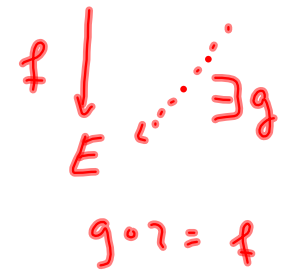
every module has projective resolution:

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \text{ exact}$$

P_i projective.

! Often, projective resolutions
are smaller than free resolutions.

Def: E is injective $\iff 0 \rightarrow A \xrightarrow{\sim} B$

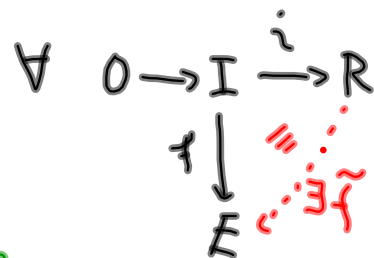


4.10 E injective $\iff \text{Hom}(-, E)$ is exact

Analogous to 4.5.

[Baer Criterion]

E is injective \iff



with I a left ideal of R

Proof " \implies " \checkmark

" \impliedby " Given $0 \rightarrow A \xrightarrow{\sim} B$

$$\begin{array}{ccc} & \nearrow \dots & \\ h \downarrow & & \\ E & \xrightarrow{\dots} & \end{array}$$

wlog inclusion

$$\mathcal{E} = \left\{ (A', h') : A \subseteq A' \subseteq B \text{ and } h' \text{ extends } h \right\}$$

$(A', h') \in \mathcal{E}$. \mathcal{E} has partial order

$$(A', h') \leq (A'', h'') \iff A' \subseteq A''$$

check Zorn's lemma applies h'' extends h'

\exists maximal element $(A_0, h_0) \in \mathcal{E}$
WANT $A_0 = B$

Assume (for contr.) $\exists x \in B \setminus A_0$

Let $I = \{r \in R : rx \in A_0\}$ left ideal of R

Define $f : I \rightarrow E$

$$f(r) = h_0(rx)$$



By assumption

$\exists \tilde{f} : R \rightarrow E$ extending f .

$$A_0 \subsetneq A_1 := A_0 + Rx \xrightarrow{h_1} E$$

$$h_1(a_0 + rx) := h_0(a_0) + r\tilde{f}(1)$$

check this will define ... DIYS.

This extends h_0

$$\text{So } (A_0, h_0) \subsetneq (A_1, h_1) \in \mathcal{E}$$

Contradiction to maximality.

So $A_0 = B$, done.

Digression: R fdim K -algebra,
 finite dim. modules \rightsquigarrow can use duality.

$$D = \text{Hom}_K(-, K)$$

(1) $V \in K\text{-mod}$ (finite dim)

$$\Rightarrow D(D(V)) \xleftarrow{\sim} V$$

$$\text{ev}_V \quad \leftarrow \quad \sim$$

$$\text{ev}_V: D(V) \rightarrow K \quad \text{evaluation.}$$

$$f \rightarrow f(v)$$

$\dim V < \infty \Rightarrow \text{trivial isomorphism.}$

R -modules: $D(M_R)$ is a left module ($(\sim f)(m) = f(mv)$)
 $D: \text{mod } R \xrightarrow{\sim} R\text{-mod}$

$$D \circ D \xleftarrow{\sim} \text{id on mod } R$$

(on R -mod)

(2) $D: \text{Hom}(X_R, Y_R) \rightarrow \text{Hom}(D(Y), D(X))$
 is iso (X, Y f-dim).

$\Rightarrow P_R$ projective $\Rightarrow D(R_R)$ is an
 injective left module.

$$0 \rightarrow I \rightarrow R$$

$$\downarrow \quad \text{---} \quad \downarrow$$

$$D(P_R)$$

$$0 \leftarrow DI \leftarrow DR$$

$$\uparrow \quad \text{---} \quad \uparrow$$

$$D^2/P_R$$

$$\uparrow$$

$$R$$

$$\uparrow$$

$$P_R$$

projection