Problem sheets:

deadline 15 April 2012

[~ 4 sheets in total.
Do $\gtrsim$ 50% (correct)].

recommend: do problems as we go along.

4.6

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & D & \xrightarrow{p_X} & X & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle p_B} & & \big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

$D = \{ (b,x) \in B \oplus X : \beta(b) = \alpha(x) \}$

$p_X(b,x) = x \qquad p_B(b,x) = b.$

$i(a) := (a, 0)$ i is 1-1, p_X is onto

LHS square:

$$\begin{array}{ccc} a & \longrightarrow & (a,0) \\ \downarrow & & \downarrow \\ a & \longrightarrow & a \end{array}$$

RHS square

$$\begin{array}{ccc} (b,x) & \longrightarrow & x \\ \downarrow & & \downarrow \\ b & \longrightarrow & \beta(b) = \alpha(x) \end{array}$$

$\operatorname{Im} i \stackrel{?}{=} \operatorname{Ker} p_X$

$\supseteq$ Let $p_X(b,x) = 0$, so $x = 0$

$\Rightarrow \alpha p_X(b,0) = \alpha(0) = 0$
$\quad\;\; \|$
$\beta p_B(b,0) = \beta(b)$

$b \in \operatorname{Ker} \beta = A$

$(b,0) = i(b)$

$\in \operatorname{Im} i$

$\subseteq$ easy.

P projective :

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists \gamma"'] \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

$\Longleftrightarrow$ Hom$(P, -)$ is exact.

4.7 : P projective $\Longleftrightarrow$ $\forall\ 0 \to A \to B \xrightarrow{\pi} P \to 0$ is split.

"$\Longrightarrow$"

$$\begin{tikzcd} & & & P \arrow[dl, dotted, "j"'] \arrow[d, "1_P"] \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, "p"] & P \arrow[r] & 0 \end{tikzcd}$$

$p \circ j = 1_P$

in §2 : Sequence is split

$\Longleftarrow$

Let
$$\begin{tikzcd} & P \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

SES $0 \to A \xrightarrow{\iota} B \xrightarrow{\beta} C \to 0$, $A = \ker\beta$

Apply 4.6

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "id"] & D \arrow[r, "p_P"] \arrow[d, "p_B"] & P \arrow[r] \arrow[d, "\alpha"] \arrow[l, dotted, bend left, "j"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd}$$

By hypothesis, top row is split $p_P \circ j = 1_P$

Try $\gamma = p_B \circ j : P \to B$

$$\pi \circ p_B \circ j = \alpha\, p_P\, j = \alpha\, 1_P = \alpha$$

4.8 a) $P = Q \oplus X$, P projective

$$\begin{tikzcd} & P \arrow[d, shift right, "p"'] \arrow[dl, dotted, "\exists \gamma"'] \\ & Q \arrow[u, shift right, "\lambda"'] \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

$p \circ \lambda = 1_Q$

P projective: $\exists \gamma$ with $\beta \circ \gamma = \alpha p$

$\Rightarrow \beta \circ \gamma \circ \lambda = \alpha p \lambda = \alpha 1_Q = \alpha$.

So Q is projective.

(b) $\Leftarrow$ F free $\Rightarrow$ projective 4.3 $\Rightarrow$ any summand is projective by a)

$\Rightarrow$ Assume P is projective.

4.2: $\exists$ free module F and an epi $F \longrightarrow P \to 0$

$\Rightarrow$ have $0 \to A \longrightarrow F \longrightarrow P \to 0$ SES

By 4.7 this is split $\therefore$ $F \simeq P \oplus A$.

1

<u>Ex</u> $R = k\langle g\rangle$ $\langle g\rangle$ group, $g^2 = 1 \neq g$

assume char $k \neq 2$

$$\left(\tfrac{1}{2}(1 \pm g)\right)^2 = \tfrac{1}{4}(1 + g^2 \pm 2g) = \tfrac{1}{4}(2(1 \pm g))$$

in the group algebra

$1 = e_0 + e_1$, $e_0 e_1 = 0$ $= \tfrac{1}{2}(1 \pm g)$

By Linear Algebra

$R = Re_0 \oplus Re_1$, Re_i is 1-dim

${}_R R$ is free, so Re_i is projective.

not free: R is the smallest $\neq 0$ projective mod

$\dim_k R = 2$, $\dim_k Re_i = 1$ $Re_i \not\cong R$

$R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

as R-modules

$\mathbb{Z}/2\mathbb{Z}$ is projective as R-module

not free

4.9 Application:

every module has projective resolution:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \text{ exact}$$

P_i projective.

! Often, projective resolutions are <u>smaller</u> than free resolutions.

Def: E is injective $\iff$
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dotted, "\exists g"] \\ & E & \end{tikzcd}$$
$g \circ i = f$

<u>4.10</u> E injective $\iff$ $\mathrm{Hom}(-,E)$ is exact

Analogous to 4.5.

[Baer Criterion]
E is injective $\iff$ $\forall$
$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[dl, dotted, "\exists \tilde{f}"] \\ & E & \end{tikzcd}$$
with I a left ideal of R

Proof "$\Longrightarrow$" ✓

"$\Longleftarrow$" Given
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "h"'] & B \arrow[dl, dotted, "\exists ?"] \\ & E & \end{tikzcd}$$
wlog i inclusion

$$\mathcal{E} = \{ (A', h') : A \leq A' \leq B \text{ and } h' \text{ extends } h \}$$

$(A,h) \in \mathcal{E}$. $\mathcal{E}$ has partial order

$(A',h') \leq (A'',h'') \iff A' \leq A''$, h'' extends h'

check Zorn's lemma applies

$\exists$ maximal element (A_0, h_0) in $\mathcal{E}$

WANT $A_0 = R$

Assume (for contra) $\exists\ x \in B \setminus A_0$

Let $I = \{ r \in R : rx \in A_0 \}$ left ideal of R

Define $f : I \longrightarrow E$

$$f(r) = h_0(rx)$$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "h"'] & A_0 \arrow[r, hook] \arrow[dl, "h_0"] & B \\ E & & \end{tikzcd}$$

By assumption

$\exists\ \tilde{f} : R \longrightarrow E$ extending f.

$A_0 \subsetneq A_1 := A_0 + Rx \xrightarrow{h_1} E$

$$h_1(a_0 + rx) := h_0(a_0) + r\tilde{f}(1)$$

check this well defined ... DIYS.

This extends h_0

So $(A_0, h_0) \subsetneq (A_1, h_1) \in \mathcal{E}$

Contradiction to maximality.

So $A_0 = B$, done.

Digression: R f.dim k-algebra,

finite dim. modules $\rightsquigarrow$ can use duality.

$D = \mathrm{Hom}_k(-, k)$

(1) $V \in k\text{-mod}$ (finite dim)

$\Rightarrow D(D(V)) \xleftarrow{\sim} V$

$ev_v \leftarrow v$

$ev_v : D(V) \to k$ evaluation.

$f \to f(v)$

$\dim V < \infty \Rightarrow$ this is isomorphism.

R-modules: $D(M_R)$ is a left module $(rf)(m) = f(mr)$

$D : \mathrm{mod}\, R \rightleftarrows R\text{-mod}$

$D \circ D \xleftarrow{\sim} \mathrm{id}$ on $\mathrm{mod}\, R$

(on R-mod)

(2) $D : \mathrm{Hom}(X_R, Y_R) \longrightarrow \mathrm{Hom}(D(Y), D(X))$

is iso (X, Y f-dim).

$\Rightarrow P_R$ projective $\Rightarrow D(R_R)$ is an injective left module.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[dl, dotted, "g"] \\
& D(P_R) &
\end{tikzcd}
\qquad
\begin{tikzcd}
0 & DI \arrow[l] & DR \arrow[l] \\
& D^2(P_R) \arrow[u] \arrow[ur, dotted, "\exists \gamma = g^*"'] &
\end{tikzcd}$$

$D^2(P_R) \cong P_R$ projective