

Proof 2.6 finished: $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$
 exact

Want:

$$M \otimes_R A \xrightarrow{1 \otimes \alpha} M \otimes_R B \xrightarrow{1 \otimes \beta} M \otimes_R C \rightarrow 0$$

Done: $1 \otimes \beta$ is onto. is exact.

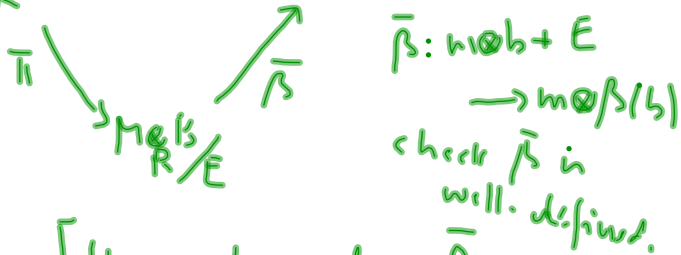
Show now $\text{Im}(1 \otimes \alpha) = \ker(1 \otimes \beta)$.

" \subseteq " $(1 \otimes \beta) \circ (1 \otimes \alpha): m \otimes a \rightarrow m \otimes \beta(a) \rightarrow m \otimes \beta(a)$
 $\beta(a) = 0 \implies 0$

Let $E = \text{Im}(1 \otimes \alpha) \subseteq \ker(1 \otimes \beta)$

Get

$$M \otimes_R B \xrightarrow{1 \otimes \beta} M \otimes_R C$$



To show

$\bar{\beta}$ is 1-1 [then: kernel of $\bar{\beta}$ is $\ker(1 \otimes \beta) / E$ in zero
 i.e. $\ker(1 \otimes \beta) = E$]

Construct map

$$\gamma: M \otimes_R C \rightarrow M \otimes_R B / E$$

To show: the following is well defined

$$\gamma(m \otimes c) := m \otimes b + E \text{ where } \beta(b) = c$$

Then $\gamma \circ \bar{\beta}(m \otimes b + E) =$

$$\gamma \circ \bar{\beta} = \text{Id} = \gamma(m \otimes \beta(b) + E) = m \otimes b + E$$

$\implies \bar{\beta}$ is 1-1.

[Ex 2.7 Hint:
apply $\text{Hom}(C, -)$. Look at dimensions
of $\text{Hom}(C, -)$'s.

Comen on abelian categories: postponed.

Motivation:

Linear algebra $(,)$ inner product:

adjoint of linear maps

$$(Av, w) = (v, A^*w)$$

Characters: $H \leq G$

$$\begin{array}{l} \chi \text{ char of } H \\ \xi \text{ char. of } G \end{array} \quad (\chi \uparrow^G, \xi \downarrow_G) = (\chi, \xi \downarrow_H) \quad H$$

Can view $(,)$ of characters
as dimension of Hom space

\leadsto generalize!

Thm 3.1 (a) ${}_R A \quad {}_S B \quad {}_R C$

$\text{Hom}_S(B, C)$ is a left R -module:

$$f: B \rightarrow C, \quad r \in R$$

$$(rf)(b) := f(rb) \quad (\text{check this works})$$

Before proving,

Special case: $R \subset S$ subring

$$\text{Take } B = {}_S S_R$$

$$B \otimes_R A = S \otimes_R A = \text{"A induced to S"} = A \uparrow^S$$

$$\text{Hom}_S(B, C) = \text{Hom}_S(S, C)$$

this is C with action of R
from $R \subset S$

3.1 becomes: i.e. "C restricted to R"

$$\text{Hom}_S(A \uparrow^S, C) \simeq \text{Hom}_R(A, C \downarrow_R) \quad \Big| = C \downarrow_R$$

Sometimes called "Frobenius Reciprocity".

Proof 3.1 a)

To define $\tau : \text{Hom}_S(B \otimes_R A, C) \rightarrow \text{Hom}_R(A, \text{Hom}_S(B, C))$

Let $f : B \otimes_R A \rightarrow C$

Fix $a \in A$, define $f_a : B \rightarrow C, f_a(b) := f(b \otimes a)$

f_a is an S -module hom:

$$\begin{aligned} f_a(b_1 + b_2) &= f(b_1 + b_2 \otimes a) \\ &= f(b_1 \otimes a + b_2 \otimes a) \\ &= f(b_1 \otimes a) + f(b_2 \otimes a) \\ &= f_a(b_1) + f_a(b_2) \\ f_a(sb) &= f(sb \otimes a) \\ &= f(s(b \otimes a)) \\ &= sf(b \otimes a) = sf_a(b) \end{aligned}$$

Define $\bar{f} : A \rightarrow \text{Hom}_S(B, C)$
 $a \mapsto f_a$

Check \bar{f} is an R -module hom $\therefore \bar{f} \in \text{RHS}$

Def: $\tau(f) := \bar{f}$.

Check τ is an additive hom. $\tau(f_1 + f_2) = \bar{f}_1 + \bar{f}_2$.

Want: τ is iso.

Construct inverse.



Let $g : A \rightarrow \text{Hom}_S(B, C)$

Def $g' : B \otimes_R A \rightarrow C$ by
 $g'(b \otimes a) := g(a)(b)$

check well-defined...

Define $\phi(g) = g'$

Check $\phi \circ \tau = \text{id}$ and $\tau \circ \phi = \text{id}$.

3.1 in terms of functors:

$$F = B \otimes_R - \quad G = \text{Hom}_S(B, -)$$

3.1 becomes

$$\text{Hom}_S(FA, C) \cong \text{Hom}_R(A, GC)$$

Looks formally same as "adjoint linear maps"

Two more examples: K field

$$U: \begin{array}{ccc} K\text{-Mod} & \longrightarrow & \mathcal{S} \\ \text{v-spaces} & & \text{sets} \end{array} \quad \begin{array}{l} \text{forgetful} \\ \text{functor} \end{array}$$

$$\longleftarrow F$$

X set, $F(X) :=$ the K -vector space with basis X

$$X \xrightarrow{f} Y \quad F(X) \xrightarrow{F(f)} F(Y)$$

the linear map defined on basis $x \rightarrow f(x) \in Y \subseteq F(Y)$

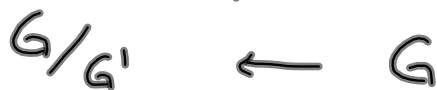
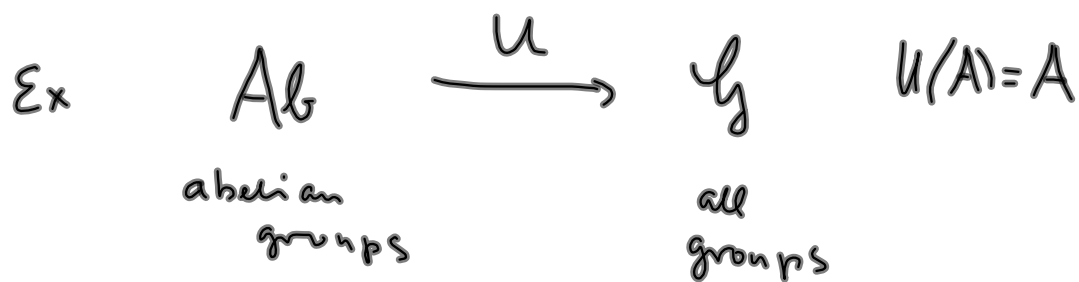
can show

$$\tilde{\tau}: \text{Hom}_K(FX, V) \xrightarrow{\sim} \text{Hom}_{\mathcal{S}}(X, U(V))$$

$$\alpha: \overset{\cup}{FX} \longrightarrow V$$

$$\tilde{\tau}(\alpha): X \longrightarrow U(V)$$

$$\text{i.e. } \tilde{\tau}(x) = \alpha|_x \quad \begin{array}{l} \text{as set} \\ \text{set } X \end{array}$$



$\exists \tau: \text{Hom}_{Ab}(\mathbb{F}(G), A) \xrightarrow{\sim} \text{Hom}_{\mathcal{G}}(G, U(A))$
 (adjoint)