4. Free, projective, injective, flat modules

Definition $F \in R-$Mod is **free** $\Leftrightarrow$ F is isomorphic to a sum of copies of R. That is $F \cong \sum_{j \in J} F_j$ where $F_j \cong R$.

$F_j \cong R$ $\Leftrightarrow$ $F_j = Rx_j$ for some $x_j \in F_j$ with $rx_j = 0$ if and only if $r = 0$.

If so then $X := \{x_j : j \in J\}$ is called an R-basis of F. Every $x \in F$ has a *unique* expression $x = \sum r_i x_i$ with $r_i \in R$ and almost all $r_i = 0$.

Universal property:

4.1 Theorem *Let F be free with basis X. Given any module B and any function f from the the set X to B, there is a unique R-module homomorphism $\tilde{f} : F \to B$ extending f.*

Proof

4.2 Lemma *Every* $M \in R - \mathrm{Mod}$ *is a homomorphic image of a free module.*

Proof Take F free, with basis indexed by all elements of M, say $X = \{x_m : m \in M\}$ (such F exists...).

Define $f : X \to M$ (a map on a set) by $f(x_m) := m$. This is already onto.
Take $\tilde{f} : F \to M$ be the R-homomorphism extending f.

DEF A free resolution of M is an exact sequence

$$\ldots \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$$

in which each F_i is a free R-module.

- **Every M has a free resolution:**

4.3 Proposition *Suppose* $B \xrightarrow{\beta} C \to 0$ *is exact. If* F *is free and* $\alpha : F \to C$ *is any* R*-module homomorphism then there exists* $\gamma : F \to B$ *with* $\alpha = \beta\gamma$*.*

[γ not unique in general.]

Proof Let $X = \{x_j\}_{j \in J}$ be an R- basis for F.

For each j there is some $b_j \in B$ such that $\beta(b_j) = \alpha(x_j)$.

Then there is a map (on sets) $\phi : X \to B$ such that $\phi(x_j) = b_j$ [`by axiom of choice`]. By 4.1 there is an R-module homomomorphism $\gamma : F \to B$ with $\gamma(x_j) = \phi(x_j)$ for all j.

$\alpha = \beta\gamma$: check on basis X.

4.4 Corollary *F free $\Rightarrow$ the functor* $\text{Hom}_R(F,-)$ *is exact.*

Proof Left exact by 2.5.

To show right exact. Let $B \xrightarrow{\beta} C \to 0$ be exact, to show

$\beta_* : \text{Hom}(F, B) \to \text{Hom}(F, C)$ is onto.

Let $f \in \text{Hom}(F, C)$, by 4.3 there is $g \in \text{Hom}(D, B)$ with $\beta \circ g = f$. By definition $\beta \circ g = \beta_*(g)$.

Question For which modules P is $\text{Hom}_R(P,-)$ exact?

For which E is $\text{Hom}_R(-,E)$ exact?

Projective modules

DEF A module P is **projective** $\Leftrightarrow$ for any surjection $\beta : B \to C$ and any $\alpha : P \to C$ there is $\gamma : P \to B$ such that $\alpha = \beta\gamma$.

Example Any free module is projective (by 4.3).

4.5 Theorem *P is projective* $\Leftrightarrow$ $\mathrm{Hom}(P, -)$ *is exact.*

Proof

4.6 Lemma Given a SES $0 \to A \xrightarrow{\iota} B \xrightarrow{\beta} C \to 0$ and $\alpha : X \to C$. This can be completed to a commutative diagram with exact rows (with D the pull-back)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & D & \xrightarrow{p_X} & X & \longrightarrow & 0 \\ & & \text{Id}\big\downarrow & & p_B\big\downarrow & & \alpha\big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

Proof

Exercise Formulate and prove the dual of this. [With push-out]

4.7 Lemma P is projective $\Leftrightarrow$ every SES $0 \to A \to B \to P \to 0$ is split.

4.8 Theorem (a) Every summand of a projective is projective.
(b) P is projective $\Leftrightarrow$ P is isomorphic to a summand of a free module.

Proof of 4.8 (a)

(b) $\Rightarrow$ By 4.2 there is a SES

$$0 \rightarrow A \xrightarrow{\iota} F \rightarrow P \rightarrow 0$$

with F free. By 4.7, this is split. By Lemma (Ch 2) $F \cong P \oplus A$.

$\Leftarrow$ 4.3 and (a)

Examples (1) $R = KG$, where $G = \langle g \rangle$, and char $(K) \neq 2$. Then $1_R = e_0 + e_1$ where

$$e_0 = (1/2)(v_1 + v_g), \quad e_1 = (1/2)(v_1 - v_g)$$

These are idempotent and $e_0 e_1 = 0$. Therefore

$$R = Re_0 \oplus Re_1$$

as R-modules.
Then Re_i is projective but not free. (Too small, the smallest projective module is R and $\dim R = 2$ but $\dim Re_i = 1$).

(2) $R = \mathbb{Z}/6\mathbb{Z}$, then (by 'Chinese Remainder Theorem')
$R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ as R-modules. So $\mathbb{Z}/2\mathbb{Z}$ is projective as R-module. But it is not free (too small).

4.9 Proposition Every module is the homomorphic image of a projective module.

Proof M is a homomorphic image of a free module by 4.2. A free module is projective.

Injective modules

DEF An R-module E is **injective** if for every exact sequence

$$0 \to A \xrightarrow{\iota} B$$

and every $f : A \to E$ there is $g : B \to E$ such that $g\iota = f$.

4.10 Theorem E is injective $\Leftrightarrow$ the functor Hom$(-, E)$ is exact.

[analogous to 4.5.]

To get examples, use Baer!

4.11 Theorem (Baer criterion) Let $E \in R-$ Mod. Then E is injective $\Leftrightarrow$

- every homomorphism $f : I \rightarrow E$ for I a left ideal of R can be extended to $\tilde{f} : R \rightarrow E$.

Proof

Example Let R be a finite-dimensional algebra over some field K. Let P be a finite-dimensional projective right R-module. Then

$$D(P) := \mathrm{Hom}_K(P, K)$$

is a left R-module, It is injective.

Proof (! thanks to Baer)

4.12 Proposition E is injective $\Leftrightarrow$ every SES $0 \to E \to B \to C \to 0$ is split.

Aim Every module is a submodule of some injective module.

Much more work than the 'dual' version for projective modules. We'll give a sketch.

4.13 Lemma A module E is injective if and only if every SES

$$0 \to E \to B \to C \to 0$$

with C cyclic, splits.

Proof

DEF Let M be some R-module, $m \in M$.
(a) m is divisible by $r \in R$ if $m = rm'$ for some $m' \in M$.
(b) M is divisible $\Leftrightarrow$ every $m \in M$ is divisible by every non-zero-divisor in R.

Example $(\mathbb{Q}, +)$ is a divisible $\mathbb{Z}$-module.

4.14 Theorem Every injective module E is divisible.

Proof Let $m \in E$ and $r_0 \in R$ be a non-zero divisor. Define $f : Rr_0 \to E$ by

$$f(rr_0) := rm$$

(well-defined!) Since E injective $\Rightarrow$ $\exists$ $g : R \to E$ extending f. Hence $m = f(r_0) = g(r_0) = r_0 g(1)$.

Exercise Show that
(a) a quotient of a divisible module is divisible.
(b) the sum of divisible modules is divisible.

4.15 Theorem Assume R is a PID. Then an R-module D is injective $\Leftrightarrow$ D is divisible.

Proof

4.16 Theorem Every $\mathbb{Z}$-module G ($=$ 'abelian group') can be embedded into an injective $\mathbb{Z}$-module.

Proof G is a quotient of a free $\mathbb{Z}$ module, say $G = F/S$ and $F = \sum_j \mathbb{Z}$. Embed each copy of $\mathbb{Z}$ into $\mathbb{Q}$, then $G \subseteq (\sum_j \mathbb{Q})/S$.
Apply Exercise and the previous Theorem.

4.17 Theorem Suppose D is a divisible abelian group, and R any ring then $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective left R-module.

(The proof this needs adjoint functors. Exercise).

So eg $\text{Hom}_{\mathbb{Z}}(R, \mathbb{Q})$ is an injective left R-module.

4.18 Theorem Every $M \in R-$ Mod can be embedded into an injective module.

Proof Regard M as abelian group. By 4.16 we have a 1-1 $\mathbb{Z}$-module homomorphism, say $\iota : M \to D$ with D is an injective $\mathbb{Z}$-module.

For $m \in M$, define $f_m : R \to M$ by $r \mapsto rm$ Define then $\varphi : M \to \text{Hom}_{\mathbb{Z}}(R, D)$ by

$$m \mapsto \iota \circ f_m$$

Check that this is R-module hom and that it is 1-1.

Flat modules

Definition A right module $\in$ Mod $-R$ is **flat** if the functor $B \otimes_R (-)$ is exact. [Similarly for left modules]

4.19 Theorem R is flat (both as left and as right R-module).

Proof

4.20 Theorem Let $\{B_j : j \in J\}$ be a family of right R-modules. Then $\sum_j B_j$ is flat $\Leftrightarrow$ each B_j is flat.

Proof

Corollary Every projective module is flat.