

Homological algebra, HT12, sheet 2

If not specified otherwise, R is a ring, and A, B, \dots are R -modules.

Qu. 1 Assume k is a field, and \mathcal{C} is the category of commutative k -algebras (where the morphisms are algebra homomorphisms). Show that $A \otimes_k B$ is the coproduct of A and B in \mathcal{C} .

2. Let $U : R\text{-Mod} \rightarrow \mathbf{Sets}$ be the forgetful functor, and let $F : \mathbf{Sets} \rightarrow R\text{-Mod}$ be the ‘free functor’. That is, FX is the free R -module with basis X , and if $f : X \rightarrow Y$ is a set map then Ff is the map as in 4.1 of the lecture.

Prove that F is indeed a functor. Prove also that (F, U) is an adjoint pair.

3. Let (F, G) be some adjoint pair of functors,

$$\tau = \tau_{A,C} : \text{Hom}_{\mathcal{C}}(FA, C) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(A, GC)$$

the adjoint isomorphism, and let $\eta : \text{Id} \rightarrow GF$ be the unit and $\varepsilon : FG \rightarrow \text{Id}$ be the counit (see lecture notes).

(a) Show that if $f : FA \rightarrow C$ then $\tau(f) = G(f) \circ \eta_A$.

(b) Show that if $g : A \rightarrow GC$ then $\tau^{-1}(g) = \varepsilon_C \circ F(g)$.

4. If you have done qu 2 then you could try to write out details of the proof that if (F, G) is an adjoint pair of functors then F preserves direct limits.

5. What can you say about the pushout of two homomorphisms $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$, one of which is the zero map? What can you say about the pushout of f and g if one of them is an isomorphism?

6. Assume $I = \mathbb{N}$ with the natural order (so that I is a quasi-ordered set). Assume $A_1 \subset A_2 \subset \dots$ is an ascending sequence of submodules of a module X . Prove that $\varinjlim A_i \cong \bigcup_{i=1}^{\infty} A_i$.

7. Let p be a prime number, and let $A_n = \mathbb{Z}/p^n\mathbb{Z}$, the \mathbb{Z} -module (or cyclic group) with p^n elements. Let $I = \mathbb{N}$ with the natural order.

(a) Check that the maps $\phi_n : A_n \rightarrow A_{n+1}$ where $\phi_n(x + p^n\mathbb{Z}) = px + p^{n+1}\mathbb{Z}$ define an inverse system. Show that $\varinjlim A_n$ is isomorphic to the group C_{p^∞} of complex numbers which are p^n 'th roots of unity.

- (b) Check that the maps $\psi_n : A_n \rightarrow A_{n-1}$ where $\psi_n(x + p^n\mathbb{Z}) = x + p^{n-1}\mathbb{Z}$ define a direct system, even of ring homomorphism. Let $\mathbb{Z}_p := \varprojlim A_n$. Check that this is a subring of $\prod_{n \geq 1} A_n$. [This is the ring of p -adic integers].
- (c) Are $\varinjlim A_n$ and $\varprojlim A_n$ isomorphic as abelian groups?

8. Let $R = \mathbb{Z}$, and consider the pull back W of the diagram

$$\begin{array}{ccc} W & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \downarrow & & f \downarrow \\ \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/4\mathbb{Z} \end{array}$$

of \mathbb{Z} -modules (or abelian groups), with homomorphisms defined as $f(x + 2\mathbb{Z}) := 2x + 4\mathbb{Z}$ and where $\pi(z) = z + 4\mathbb{Z}$. Then W is a \mathbb{Z} module (or abelian group). Can you identify it (ie find its isomorphism type).

9. Assume $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is a SES, and $f : A \rightarrow X$ is some R -module homomorphism. Show that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & f \downarrow & & f' \downarrow & & 1_C \downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow{\alpha'} & L & \xrightarrow{\beta'} & C & \longrightarrow & 0 \end{array}$$

in which the LHS square is a push-out.