

# GALOIS SECTIONS FOR ABELIANIZED FUNDAMENTAL GROUPS

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*With an appendix by E. V. Flynn*

## 1. INTRODUCTION

Let  $k$  be a perfect field with fixed algebraic closure  $\bar{k}$ , and  $X$  a geometrically integral separated scheme of finite type over  $k$ . Choose a geometric point  $\bar{x} : \text{Spec } \bar{k} \rightarrow X$ . One then has the well-known exact sequence of profinite groups

$$(1.1) \quad 1 \rightarrow \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}(\bar{k}|k) \rightarrow 1,$$

where  $\pi_1(X, \bar{x})$  denotes the étale fundamental group of  $X$ , and  $\bar{X}$  stands for the base change  $X \times_k \bar{k}$ .

Each  $k$ -rational point  $\text{Spec } k \rightarrow X$  determines a section of the structure map  $X \rightarrow \text{Spec } k$ . As the étale fundamental group is functorial for morphisms of pointed schemes, taking the induced map on fundamental groups defines a map

$$X(k) \rightarrow \{\text{continuous sections of } \pi_1(X, \bar{x}) \rightarrow \text{Gal}(\bar{k}|k)\} / \sim,$$

where two sections are equivalent under the relation  $\sim$  if they are conjugate under the action of  $\pi_1(\bar{X}, \bar{x})$ .

According to the projective case of the famous Section Conjecture of Grothendieck formulated in [9], the above map should be a bijection when  $k$  is a finitely generated field over  $\mathbf{Q}$  and  $X$  is a smooth projective curve of genus at least 2. Here injectivity is easy to prove and was known to Grothendieck; the hard and widely open part is surjectivity. Our main motivation for writing the present paper is the following variant of Grothendieck's conjecture.

**Conjecture 1.1.** *A smooth projective curve  $X$  of genus at least 2 over a number field  $k$  has a  $k$ -rational point if and only if the map  $\pi_1(X, \bar{x}) \rightarrow \text{Gal}(\bar{k}|k)$  has a continuous section.*

As several people have reminded us, together with the theorem of Faltings (*quondam* Mordell's Conjecture) this seemingly weaker statement actually implies the surjectivity part of Grothendieck's Section Conjecture over a number field (see [13], Lemma 1.7, itself based on an idea of Tamagawa [26]).

Oddly enough, until the recent preprint [25] of J. Stix there seems to have been no examples where the statement of Conjecture 1.1 holds in a nonobvious way, i.e. where  $X(k) = \emptyset$  and  $\pi_1(X, \bar{x}) \rightarrow \text{Gal}(\bar{k}|k)$  has no continuous section. This may be because, as we shall see below, verifying that sections do *not* exist is by no means straightforward. In the examples of Stix the curve  $X$  has no points over some non-archimedean completion of  $k$ . Thus the question remained whether there are curves of genus at least 2 where  $\pi_1(X, \bar{x}) \rightarrow \text{Gal}(\bar{k}|k)$  has no continuous section and at the same time  $X$  has points everywhere locally, or in other words it is a counterexample to the Hasse principle for rational points.

In this paper we answer this question via investigating sections of the abelianized fundamental exact sequence of  $X$ . This is the short exact sequence

$$(1.2) \quad 1 \rightarrow \pi_1^{\text{ab}}(\bar{X}) \rightarrow \Pi^{\text{ab}}(X) \rightarrow \text{Gal}(\bar{k}|k) \rightarrow 1$$

obtained from (1.1) by pushout via the natural surjection  $\pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1^{\text{ab}}(\bar{X})$ , where  $\pi_1^{\text{ab}}(\bar{X})$  is the maximal abelian profinite quotient of  $\pi_1(\bar{X}, \bar{x})$ . Of course, if (1.1) has a continuous section, then so does (1.2).

Many of our considerations will be valid in arbitrary dimension. In dimension greater than one, however, a slight complication arises from torsion in the Néron-Severi group. To deal with it, we introduce yet another pushout of exact sequence (1.1). Recall (e.g. from [12], Lemma 5) that for a smooth projective geometrically integral  $k$ -scheme  $X$  there is an exact sequence

$$(1.3) \quad 0 \rightarrow S \rightarrow \pi_1^{\text{ab}}(\bar{X}) \rightarrow T(\text{Alb}_X(\bar{k})) \rightarrow 0$$

of  $\text{Gal}(\bar{k}|k)$ -modules, where  $\text{Alb}_X$  denotes the Albanese variety of  $X$ , the notation  $T(\text{Alb}_X(\bar{k}))$  means its full Tate module, and  $S$  is the finite group dual to the torsion subgroup of the Néron-Severi group  $NS(\bar{X})$  of  $\bar{X}$ . Taking the pushout of exact sequence (1.2) by the surjection  $\pi_1^{\text{ab}}(\bar{X}) \rightarrow T(\text{Alb}_X(\bar{k}))$  we obtain an extension

$$(1.4) \quad 1 \rightarrow T(\text{Alb}_X(\bar{k})) \rightarrow \Pi \rightarrow \text{Gal}(\bar{k}|k) \rightarrow 1$$

of profinite groups. In the case when  $NS(\bar{X})$  is torsion-free (e.g. for curves or abelian varieties) this sequence is the same as (1.2).

Recall also (e.g. from [7], Theorem 4.1) that by the theory of Albanese varieties over arbitrary perfect fields there is a canonical  $k$ -torsor  $\text{Alb}_X^1$  under  $\text{Alb}_X$ , characterized by the universal property that every morphism from  $X$  to a  $k$ -torsor under an abelian variety factors uniquely through  $\text{Alb}_X^1$ . We may finally state:

**Theorem 1.2.** *Let  $X$  be a smooth projective geometrically integral variety over a perfect field  $k$ .*

The map  $\Pi \rightarrow \text{Gal}(\bar{k}|k)$  has a continuous section if and only if the class of  $\text{Alb}_X^1$  lies in the maximal divisible subgroup of the group  $H^1(k, \text{Alb}_X)$  of isomorphism classes of  $k$ -torsors under  $\text{Alb}_X$ .

We remind the reader that the maximal divisible subgroup of an abelian group may be smaller than the subgroup of infinitely divisible elements (the latter is not always a divisible subgroup). Thus the condition of the theorem is stronger than just requiring  $\text{Alb}_X^1$  to be infinitely divisible.

The condition of the theorem is related to zero-cycles on  $X$ . Namely, it follows from Theorem 4.2 of [7] that the *triviality* of the torsor  $\text{Alb}_X^1$  is equivalent to the surjectivity of the map  $CH_0(\bar{X})^{\text{Gal}(\bar{k}|k)} \rightarrow \mathbf{Z}$ , i.e. the existence of a Galois-invariant zero-cycle class of degree 1 on  $\bar{X}$ . In some cases the latter condition is equivalent to the surjectivity of the map  $CH_0(X) \rightarrow \mathbf{Z}$ , i.e. the existence of a zero-cycle of degree one on  $X$ . This is the situation, for instance, for curves over a number field having points everywhere locally (see [18], Proposition 2.5 or [7], Proposition 3.2), or for curves of odd genus over a  $p$ -adic field ([15], Theorem 7 c).

We may thus view Theorem 1.2 as a link between the existence of a section for (1.4) and the existence of a Galois-invariant zero-cycle class of degree 1 on  $\bar{X}$ . Over special fields we can say more about the relation of these two conditions. When  $k$  is a finite extension of  $\mathbf{Q}_p$ , we shall see that the two conditions are equivalent for curves of genus  $g$  with  $p$  prime to  $g - 1$  (a fact essentially going back to Lichtenbaum [15]). On the other hand, we shall give examples of other curves and higher-dimensional varieties where (1.4) splits but there is no Galois-invariant zero-cycle class. This yields the somewhat surprising fact that for curves over a  $p$ -adic field the sections of exact sequence (1.2) do not detect zero-cycles of degree 1 in general.

Over number fields we understand the situation less well but, inspired by work of Bashmakov [2], we at least give examples where  $H^1(k, \text{Alb}_X)$  has trivial maximal divisible subgroup, and therefore the splitting of (1.4) is equivalent to the existence of a Galois-invariant zero-cycle class of degree 1 on  $\bar{X}$ . In particular, we shall point out the following instance of the failure of a local-global principle for sections of exact sequence (1.1):

**Proposition 1.3.** *Let  $X$  be a smooth projective curve over  $\mathbf{Q}$  whose Jacobian is isogenous over  $\mathbf{Q}$  to a product of elliptic curves each of which has finite Tate-Shafarevich group and infinitely many  $\mathbf{Q}$ -points. Assume moreover that  $X$  has points everywhere locally but no  $\mathbf{Q}$ -rational divisor class of degree 1. Then (1.1) has sections everywhere locally but not globally.*

As Victor Flynn shows in the appendix, curves of genus 2 satisfying the assumptions of the proposition exist.

In the last section we shall discuss the relation of the splitting of (1.4) to the elementary obstruction of Colliot-Thélène and Sansuc for the existence of rational points on varieties ([3], [27]).

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## 2. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. Denote the Galois group  $\text{Gal}(\bar{k}|k)$  by  $\Gamma$ . The starting observation is that the extension (1.4) of topological groups gives rise to a class  $[\Pi]$  in Tate's continuous cohomology group  $H_{\text{cont}}^2(\Gamma, T(\text{Alb}_X)(\bar{k}))$ , defined by means of continuous cocycles. It is the trivial class if and only if the projection  $\Pi \rightarrow \Gamma$  has a continuous section.

There is a product decomposition

$$H_{\text{cont}}^2(\Gamma, T(\text{Alb}_X)(\bar{k})) \cong \prod_{\ell} H_{\text{cont}}^2(\Gamma, T_{\ell}(\text{Alb}_X)(\bar{k}))$$

where  $\ell$  runs over the set of all primes. Under this decomposition  $[\Pi]$  corresponds to a system of classes  $[\Pi_{\ell}] \in H_{\text{cont}}^2(\Gamma, T_{\ell}(\text{Alb}_X)(\bar{k}))$ , each of which is the class of the extension

$$(2.5) \quad 1 \rightarrow T_{\ell}(\text{Alb}_X(\bar{k})) \rightarrow \Pi_{\ell} \rightarrow \Gamma \rightarrow 1$$

obtained from (1.4) by pushout via the natural projection  $T(\text{Alb}_X(\bar{k})) \rightarrow T_{\ell}(\text{Alb}_X(\bar{k}))$ . Now Theorem 1.2 follows from applying the following proposition for each prime number  $\ell$ .

**Proposition 2.1.** *The class  $[\Pi_{\ell}] \in H_{\text{cont}}^2(\Gamma, T_{\ell}(\text{Alb}_X)(\bar{k}))$  is trivial if and only if the class of  $\text{Alb}_X^1$  lies in the maximal  $\ell$ -divisible subgroup of the group  $H^1(k, \text{Alb}_X)$ .*

For some of our considerations it will be more convenient to work with Jannsen's continuous étale cohomology groups introduced in [11]. The coefficients in Jannsen's theory are inverse systems  $(\mathcal{F}_n)$  of étale sheaves indexed by the ordered set  $\mathbf{N}$  of nonnegative integers. According to Theorem 2.2 of [11], over  $\text{Spec } k$  Jannsen's groups coincide with Tate's continuous Galois cohomology groups for coefficient systems satisfying the Mittag-Leffler condition. This is the case for systems of finite Galois modules, the only ones we shall consider.

It is useful to bear in mind the exact sequences

$$(2.6) \quad 0 \rightarrow \varprojlim^1 H^{i-1}(k, \mathcal{F}_n) \rightarrow H_{\text{cont}}^i(k, (\mathcal{F}_n)) \rightarrow \varprojlim H^i(k, \mathcal{F}_n) \rightarrow 0$$

(see [11], Proposition 1.6) for all  $i > 0$ , where the left-hand-side group involving the derived inverse limit is nontrivial in general.

The class  $[\Pi_\ell] \in H_{\text{cont}}^2(\Gamma, T_\ell(\text{Alb}_X)(\bar{k}))$  we are interested in can thus be viewed as a class in Jannsen's group  $H_{\text{cont}}^2(k, (\ell^n \text{Alb}_X))$ ; here  $\ell^n A$  denotes the  $\ell^n$ -torsion part of an abelian group  $A$ . Denote by  $(\text{Alb}_X, \ell)$  the inverse system indexed by  $\mathbf{N}$  where each term is the étale sheaf  $\text{Alb}_X$  over  $\text{Spec } k$  and the maps are given by multiplication by  $\ell$ . There is an exact sequence of inverse systems of étale sheaves

$$(2.7) \quad 0 \rightarrow (\ell^n \text{Alb}_X, \ell) \rightarrow (\text{Alb}_X, \ell) \xrightarrow{\ell^n} (\text{Alb}_X, \text{id}) \rightarrow 0,$$

where the last term is a constant inverse system. Here the map  $\ell^n$  stands for the isogeny given by multiplication by  $\ell^n$ ; in particular, it is surjective. It yields a coboundary map

$$(2.8) \quad \delta_\ell : H^1(k, \text{Alb}_X) \rightarrow H_{\text{cont}}^2(k, (\ell^n \text{Alb}_X))$$

using the obvious fact that the continuous cohomology of the constant system  $(\text{Alb}_X, \text{id})$  is just the usual cohomology of  $\text{Alb}_X$ .

The key observation is now the following.

**Proposition 2.2.** *The map  $\delta_\ell$  sends the class  $[\text{Alb}_X^1] \in H^1(k, \text{Alb}_X)$  to  $[\Pi_\ell]$ .*

The proof of this fact uses a technical lemma from the literature that we copy here for the readers' convenience.

**Lemma 2.3.** ([22], Lemma 2.4.5) *Let  $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$  be a central extension of algebraic  $k$ -groups such that  $B, C$  are geometrically connected and  $A$  is finite. Let  $Y$  be a right  $k$ -torsor under  $C$ . Choose a base point  $\bar{y}_0 \in Y(\bar{k})$ , and let  $\nu : \bar{C} \rightarrow \bar{Y}$  be the isomorphism of right torsors under  $\bar{C}$  sending the neutral element to  $\bar{y}_0$ . Let  $\bar{B} \rightarrow \bar{Y}$  be the composition of  $\bar{B} \rightarrow \bar{C}$  with  $\nu$ . Then we have the extension of groups*

$$(2.9) \quad 1 \rightarrow A(\bar{k}) = \text{Aut}(\bar{B}|\bar{Y}) \rightarrow \text{Aut}(\bar{B}|Y) \rightarrow \Gamma \rightarrow 1$$

such that the induced  $\Gamma = \text{Gal}(\bar{k}|k)$ -module structure on  $A(\bar{k})$  is its usual  $\Gamma$ -module structure.

The class of the extension (2.9) in  $H^2(\Gamma, A)$  coincides with  $\partial([Y])$ , where  $\partial$  is the connecting homomorphism  $H^1(\Gamma, C) \rightarrow H^2(\Gamma, A)$ .

The proof given in [22] actually shows more: the two classes coincide 'at the cocycle level'. More precisely, if we consider the 1-cocycle  $c : \Gamma \rightarrow C(\bar{k})$  representing  $[Y] \in H^1(\Gamma, C)$  that comes from the identification of  $\bar{Y}$  with  $\bar{C}$  by means of  $\bar{y}_0$ , then in order to obtain a 2-cocycle representing the image of  $[Y]$  via  $\partial$  one has to choose a continuous map  $b : \Gamma \rightarrow B(\bar{k})$  lifting  $c$ . On the other hand, a 2-cocycle representing the class of (2.9) comes from the choice of a section of the projection  $\text{Aut}(\bar{B}|Y) \rightarrow \Gamma$ . The proof shows that the map  $\gamma \mapsto (\bar{x} \mapsto b(\gamma) \cdot \gamma(\bar{x}))$  is such a section, and the resulting 2-cocycles are the same.

**Proof of Proposition 2.2:** Fix  $n > 0$ , and denote by  $\Pi_{\ell^n}$  the quotient of  $\Pi_{\ell}$  obtained as the pushforward of (2.5) via the quotient map  $T_{\ell}(\text{Alb}_X(\bar{k})) \rightarrow {}_{\ell^n}\text{Alb}_X(\bar{k})$ . Denoting by  $\lambda$  the composite map  $\text{Alb}_{\bar{X}} \xrightarrow{\ell^n} \text{Alb}_{\bar{X}} \rightarrow \text{Alb}_X^1$ , the extension

$$1 \rightarrow {}_{\ell^n}\text{Alb}_X(\bar{k}) \rightarrow \text{Aut}(\text{Alb}_{\bar{X}} \xrightarrow{\lambda} \text{Alb}_X^1) \rightarrow \Gamma \rightarrow 1$$

identifies with

$$(2.10) \quad 1 \rightarrow {}_{\ell^n}\text{Alb}_X(\bar{k}) \rightarrow \Pi_{\ell^n} \rightarrow \Gamma \rightarrow 1.$$

Indeed, this is so for  $\text{Alb}_X^1$  in place of  $X$  by the description of the fundamental group of an abelian variety, and then the Albanese map  $X \rightarrow \text{Alb}_X^1$  induces an isomorphism of exact sequence (2.10) with the corresponding sequence for  $\text{Alb}_X^1$ .

Now apply the lemma to the exact sequence

$$(2.11) \quad 0 \rightarrow {}_{\ell^n}\text{Alb}_X \rightarrow \text{Alb}_X \xrightarrow{\ell^n} \text{Alb}_X \rightarrow 0$$

and the torsor  $\text{Alb}_X^1$  under  $\text{Alb}_X$ . It says that the class of the extension (2.10) in  $H^2(\Gamma, {}_{\ell^n}\text{Alb}_X)$  is the image of  $[\text{Alb}_X^1]$  by the coboundary map  $H^1(\Gamma, \text{Alb}_X) \rightarrow H^2(\Gamma, {}_{\ell^n}\text{Alb}_X(\bar{k}))$  coming from (2.11).

Finally, we exploit the remark made above that the identification just described holds at the level of cocycles. Making  $n$  vary we in fact obtain a coherent system of sections  $\Gamma \rightarrow \Pi_{\ell^n}$  coming from liftings of a 1-cocycle  $\Gamma \rightarrow \text{Alb}_X(\bar{k})$  representing  $[\text{Alb}_X^1]$  to the  $\ell^n$ -coverings  $\text{Alb}_{\bar{X}}(\bar{k}) \xrightarrow{\ell^n} \text{Alb}_{\bar{X}}(\bar{k})$ . The 2-cocycles  $\Gamma \times \Gamma \rightarrow {}_{\ell^n}\text{Alb}_X(\bar{k})$  induced by these sections assemble to a continuous 2-cocycle  $\Gamma \times \Gamma \rightarrow T_{\ell}(\text{Alb}_X(\bar{k}))$  that describes the class of  $\Pi$  in  $H_{\text{cont}}^2(\Gamma, T_{\ell}(\text{Alb}_X(\bar{k})))$ . By construction, it also represents the image of  $[\text{Alb}_X^1]$  by  $\delta_{\ell}$ .  $\square$

Notice that in the last argument we had to work with cocycles instead of cohomology groups because by exact sequence (2.6) the group  $H_{\text{cont}}^2(\Gamma, T_{\ell}(\text{Alb}_X(\bar{k})))$  itself is not necessarily the inverse limit of the  $H^2(\Gamma, {}_{\ell^n}\text{Alb}_X(\bar{k}))$ .

**Proof of Proposition 2.1:** By the proposition just proven the class  $[\Pi_{\ell}]$  is trivial if and only if  $[\text{Alb}_X^1]$  lies in  $\ker(\delta_{\ell})$ . It thus suffices to show that  $\ker(\delta_{\ell})$  is the maximal  $\ell$ -divisible subgroup of  $H^1(k, \text{Alb}_X)$ . To see this, note first that since  $H_{\text{cont}}^2(k, ({}_{\ell^n}\text{Alb}_X))$  has no nonzero  $\ell$ -divisible subgroup ([11], Corollary 4.9), the maximal  $\ell$ -divisible subgroup of  $H^1(k, \text{Alb}_X)$  lies in  $\ker(\delta_{\ell})$ . On the other hand, the long exact sequence coming from (2.7) shows that  $\ker(\delta_{\ell})$  equals the image of the map  $H_{\text{cont}}^1(k, (\text{Alb}_X, \ell)) \rightarrow H^1(k, \text{Alb}_X)$ . We show that this image is an  $\ell$ -divisible group, which will complete the proof.

There is a factorisation

$$H_{\text{cont}}^1(k, (\text{Alb}_X, \ell)) \rightarrow \lim_{\leftarrow} (H^1(k, \text{Alb}_X), \ell) \rightarrow H^1(k, \text{Alb}_X)$$

where the middle term is the inverse limit of the inverse system given by multiplication by  $\ell$  on the group  $H^1(k, \text{Alb}_X)$ . The first map comes from exact sequence (2.6), and it factors the map  $H^1_{\text{cont}}(k, (\text{Alb}_X, \ell)) \rightarrow H^1(k, \text{Alb}_X)$  because the inverse limit of the constant inverse system  $(H^1(k, \text{Alb}_X), \text{id})$  is of course  $H^1(k, \text{Alb}_X)$ . Now the middle term is  $\ell$ -divisible by construction, hence so is its image in  $H^1(k, \text{Alb}_X)$ .  $\square$

**Remark 2.4.** Theorem 1.2 and its proof carry over *mutatis mutandis* to the more general case when  $X$  is only assumed to be smooth and quasi-projective. The fundamental group has to be replaced by the tame fundamental group classifying covers tamely ramified over the divisors at infinity, and  $\text{Alb}_X^1$  has to be understood as Serre’s generalised Albanese torsor, which is universal for morphisms in torsors under semi-abelian varieties over  $k$  (the discussion in [7], Theorem 4.1 is in this generality). The relation with the tame fundamental group that generalises exact sequence (1.3) is explained, for instance, in [24], Proposition 4.4.

### 3. THE CASE OF A $p$ -ADIC BASE FIELD

Keeping the notations and assumptions of the previous section, recall that the *period* of  $X$  is defined as the order of the cokernel of the degree map  $CH_0(\overline{X})^{\Gamma} \rightarrow \mathbf{Z}$ , and the *index* of  $X$  as the order of the cokernel of the degree map  $CH_0(X) \rightarrow \mathbf{Z}$ . If the index is 1, one says that  $X$  has a *zero-cycle of degree one*.

It is proven in Theorem 4.2 of [7] that the period of  $X$  equals the order of the class  $[\text{Alb}_X^1]$  in  $H^1(k, \text{Alb}_X)$ . Thus by Theorem 1.2 the surjectivity of  $CH_0(\overline{X})^{\Gamma} \rightarrow \mathbf{Z}$  implies that exact sequence (1.4) has a section. The question arises whether the converse is true, or in other words, whether Galois-invariant zero-cycles of degree 1 can be detected by sections of (1.4). By Theorem 1.2 this amounts to asking:

**Question 3.1.** *Can  $[\text{Alb}_X^1]$  be a nonzero element in the maximal divisible subgroup of  $H^1(k, \text{Alb}_X)$ ?*

In this section we investigate the question in the case when  $k$  is a finite extension of  $\mathbf{Q}_p$ . The following well-known fact is crucial for our considerations:

**Fact 3.2.** *If  $A$  is an abelian variety over a  $p$ -adic field  $k$ , then  $H^1(k, A)$  is isomorphic to  $F \oplus (\mathbf{Q}_p/\mathbf{Z}_p)^r$  with  $F$  finite and  $r \geq 0$ . Indeed, by Tate’s duality for abelian varieties over  $p$ -adic fields ([19], Corollary I.3.4) the group  $H^1(k, A)$  is the  $\mathbf{Q}/\mathbf{Z}$ -dual of  $A^*(k)$ , where  $A^*$  denotes the dual abelian variety. By Mattuck’s theorem [17] the group  $A^*(k)$  has a finite index open subgroup isomorphic to  $\mathbf{Z}_p^r$  for  $r = [k : \mathbf{Q}_p] \dim A$ , whence the assertion.*

In particular, the group  $H^1(k, \text{Alb}_X)$  is the sum of a finite abelian group and a  $\mathbf{Z}_p$ -module of finite cotype, the latter property meaning

that the  $\mathbf{Q}_p/\mathbf{Z}_p$ -dual is a finitely generated  $\mathbf{Z}_p$ -module. Therefore its maximal divisible subgroup coincides with the subgroup of divisible elements, and the above question over  $p$ -adic  $k$  is equivalent to asking whether  $[\text{Alb}_X^1]$  can be a nonzero divisible element in  $H^1(k, \text{Alb}_X)$ .

A first partial answer is contained in the next proposition. It is essentially a reformulation of old results of Lichtenbaum [15], and is already implicit in [25]; the proof given here is different.

**Proposition 3.3.** *Let  $k$  be a finite extension of  $\mathbf{Q}_p$ , and  $X$  a smooth proper geometrically connected curve of genus  $g \geq 1$ . Assume that  $p$  does not divide  $(g - 1)$ . Then the map  $\Pi^{\text{ab}}(X) \rightarrow \Gamma$  has a continuous section if and only if the degree map  $CH_0(\overline{X})^\Gamma \rightarrow \mathbf{Z}$  is surjective. If  $g$  is odd, this is equivalent to saying that  $X$  has a zero-cycle of degree one.*

*Proof.* By Fact 3.2 the group  $H^1(k, \text{Alb}_X)$  is isomorphic to  $F \oplus (\mathbf{Q}_p/\mathbf{Z}_p)^r$  with  $F$  finite. Therefore a divisible class can only lie in the  $(\mathbf{Q}_p/\mathbf{Z}_p)^r$ -component, and as such must be of order a power of  $p$ . By [15], Theorem 7 a) the period of  $X$ , which is the same as the order of the class  $[\text{Alb}_X^1]$ , divides  $g - 1$ . Hence the assumption that  $p$  does not divide  $(g - 1)$  implies that the class  $[\text{Alb}_X^1]$  is divisible if and only if it is zero, i.e. when  $X$  has period 1. Now the first assertion follows from Theorem 1.2. If moreover  $g$  is odd, then according to [15], Theorem 7 c) the fact that the period is 1 entails that the index is also 1, whence the second assertion.  $\square$

We now turn to cases where the answer to Question 3.1 is negative, i.e. where the class of  $[\text{Alb}_X^1]$  is nonzero and divisible in  $H^1(k, \text{Alb}_X)$ . Such examples are easy to construct: if  $A$  is an abelian variety over the  $p$ -adic field  $k$  such that  $A(k)$  is infinite, then the structure of  $H^1(k, A)$  recalled in Fact 3.2 implies that there is a nonzero divisible class in  $H^1(k, A)$ , coming from a torsor  $X$  under  $A$ . By construction we have  $\text{Alb}_X = A$  and  $\text{Alb}_X^1 = X$ , and we are done.

The following proposition handles the more difficult task of constructing such elements for curves of genus  $> 1$ . Together with the previous discussion it shows that for curves of genus  $p + 1$  the abelian quotient of the fundamental group does not always suffice to detect zero-cycles of degree 1.

**Proposition 3.4.** *Let  $p$  be an odd prime number. There exists a curve  $Y$  of genus  $p + 1$  over  $k = \mathbf{Q}_p$  such that  $\text{Alb}_Y^1$  yields a nonzero divisible class in  $H^1(k, \text{Alb}_Y)$ . Therefore the map  $\Pi^{\text{ab}}(Y) \rightarrow \Gamma$  has a continuous section, but the degree map  $CH_0(\overline{Y})^\Gamma \rightarrow \mathbf{Z}$  is not surjective.*

The proof below is inspired from a construction used by Sharif [21]. His original aim was to construct curves over  $p$ -adic fields with given index and period.



*Proof.* To begin with, let  $E$  be the Tate elliptic curve with parameter  $p$  over  $\mathbf{Q}_p$ . By the above argument we know that there is a nonzero divisible class in  $H^1(\mathbf{Q}_p, E)$ . But we can be more specific. As  $E(\mathbf{Q}_p)$  is isomorphic to  $\mathbf{Q}_p^*/p^{\mathbf{Z}} \cong \mathbf{F}_p^* \oplus \mathbf{Z}_p$  by the multiplicative structure of  $\mathbf{Q}_p$ , we have  $H^1(\mathbf{Q}_p, E) \cong \mathbf{Z}/(p-1)\mathbf{Z} \oplus \mathbf{Q}_p/\mathbf{Z}_p$  by Tate's duality theorem recalled above. Consider a torsor  $X$  under  $E$  whose class is a divisible element of order  $p$  in  $H^1(\mathbf{Q}_p, E)$ .

We contend that  $X$  is split by the degree  $p$  cyclic and totally ramified extension  $K$  of  $\mathbf{Q}_p$ . To see this, write  $K = \mathbf{Q}_p(\pi)$ , with a uniformizing parameter  $\pi$  of  $K$  satisfying  $N_{K/k}(\pi) = p$ . The group  $E(K) = K^*/p^{\mathbf{Z}}$  is isomorphic to  $\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{F}_p^* \oplus \mathbf{Z}_p$ , and an explicit element can be written in the form  $\pi^m \zeta^i (1 + pb)$ , where  $\zeta$  is a primitive  $(p-1)$ -st root of unity and  $b \in \mathbf{Z}_p$ . The corestriction map  $E(K) \rightarrow E(\mathbf{Q}_p)$  corresponds to multiplication by  $p$  because the norm of  $\pi^m \zeta^i (1 + pb)$  is  $p^m \zeta^{pi} (1 + pb)^p$  for every  $m, i \in \mathbf{Z}$ ,  $b \in \mathbf{Z}_p$ . Since the restriction map  $H^1(\mathbf{Q}_p, E) \rightarrow H^1(K, E)$  is the dual of the corestriction map  $E(K) \rightarrow E(\mathbf{Q}_p)$  via Tate duality, we obtain that our order  $p$  class  $[X]$  is killed by restriction to  $H^1(K, E)$ .

Let  $f$  be a non-square element in the function field  $\mathbf{Q}_p(X)$  of  $X$ , and let  $Y$  be the normalization of  $X$  in  $\mathbf{Q}_p(X)(\sqrt{f})$ . Consider the commutative diagram

$$\begin{array}{ccccc} (\text{Pic } \overline{X})^\Gamma & \xrightarrow{\text{deg}} & \mathbf{Z} & \longrightarrow & H^1(\mathbf{Q}_p, \text{Pic}_X^0) \\ \downarrow & & \downarrow & & \downarrow \\ (\text{Pic } \overline{Y})^\Gamma & \xrightarrow{\text{deg}} & \mathbf{Z} & \longrightarrow & H^1(\mathbf{Q}_p, \text{Pic}_Y^0) \end{array}$$

coming from the exact sequence of  $\Gamma$ -modules

$$0 \rightarrow \text{Pic}_X^0(\overline{\mathbf{Q}_p}) \rightarrow \text{Pic } \overline{X} \rightarrow \mathbf{Z} \rightarrow 1$$

and the similar one for  $Y$ . Since the pullback map  $\text{Pic } \overline{X} \rightarrow \text{Pic } \overline{Y}$  multiplies degree by 2, the middle vertical map in the above diagram is multiplication by 2. Here  $\text{Pic}_X^0 = \text{Alb}_X$  as we are dealing with curves, and similarly for  $Y$ . As explained in the proof of [7], Theorem 4.2, the image of  $1 \in \mathbf{Z}$  in  $H^1(\mathbf{Q}_p, \text{Pic}_X^0)$  is the class of  $\text{Alb}_X^1$ , and similarly for  $Y$ . We conclude that the map  $H^1(\mathbf{Q}_p, \text{Alb}_X) \rightarrow H^1(\mathbf{Q}_p, \text{Alb}_Y)$  sends  $[X] = [\text{Alb}_X^1]$  to  $2[\text{Alb}_Y^1]$ . As  $[X]$  is a  $p$ -divisible element of order  $p$  by construction, we obtain that if the order of  $[\text{Alb}_Y^1]$  is also  $p$ , then it must be a nonzero divisible class in  $H^1(\mathbf{Q}_p, \text{Alb}_Y)$  (recall that  $p$  was assumed to be odd).

We finally show that it is possible to choose  $f$  in such a way that the order of  $[\text{Alb}_Y^1]$  is  $p$ . As in [21], for this it is sufficient to find an  $f$  whose number of ramification points is  $2mp$  with  $m$  odd (this yields a  $Y$  of genus  $mp + 1$  by the Hurwitz formula). To get an example with

$m = 1$ , we proceed as in [21]. Define a divisor  $D$  on  $\overline{X} = \overline{E}$  by

$$D := \sum_{\gamma \in \text{Gal}(K|\mathbf{Q}_p)} \gamma(1) - \sum_{\gamma \in \text{Gal}(K|\mathbf{Q}_p)} \gamma(\pi)$$

where  $\gamma(x)$  denotes the twisted action of  $\Gamma$  on  $\overline{X}(\overline{\mathbf{Q}}_p) = E(\overline{\mathbf{Q}}_p) = \overline{\mathbf{Q}}_p^*/p^{\mathbf{Z}}$  by a cocycle  $\xi$  inducing the class  $[X] \in H^1(\text{Gal}(K|\mathbf{Q}_p), E)$  (recall that  $[X]$  is split by  $K$ ). The divisor  $D$  is Galois-equivariant on  $\overline{X}$ . As  $N_{K/\mathbf{Q}_p}(\pi) = 1/p$  is trivial in  $E(\mathbf{Q}_p) = \mathbf{Q}_p^*/p^{\mathbf{Z}}$ , the same argument as in Lemma 8 of [21] shows that  $D$  is principal. Therefore we can find an  $f$  such that  $\text{div}(f) = D$ , which is the one we were looking for.  $\square$

#### 4. ET RESURREXIT BASHMAKOV

In this section we investigate Question 3.1 in the case when  $k$  is a number field. It becomes especially interesting when  $X$  has points over each completion of  $k$ , for then the class  $\text{Alb}_X^1$  lies in the Tate–Shafarevich group  $\text{III}(\text{Alb}_X)$ , a group that is conjecturally finite. In particular, a nontrivial class cannot be divisible in  $\text{III}(\text{Alb}_X)$ . By Theorem 1.2, if we knew that  $\text{III}(\text{Alb}_X)$  intersects the maximal divisible subgroup of  $H^1(k, \text{Alb}_X)$  trivially, we could conclude that exact sequence (1.4) splits if and only if  $\overline{X}$  has a Galois-invariant zero-cycle class of degree 1. As remarked in the introduction, for  $X$  of dimension 1 the latter condition is equivalent to the existence of a degree 1 divisor on  $X$ .

But here we run into a serious problem: to our knowledge it is not known whether a nonzero class in the Tate–Shafarevich group  $\text{III}(A)$  of an abelian variety  $A$  over  $k$  can be divisible in the group  $H^1(k, A)$ . This phenomenon was studied by Bashmakov in his papers [1] and [2] without deciding the issue either way. However, he gave examples where  $\text{III}(A)$ , if finite, intersects the maximal divisible subgroup of  $H^1(k, A)$  trivially. This is interesting from our point of view as it yields via Theorem 1.2 examples where the map  $\Pi \rightarrow \text{Gal}(\overline{k}|k)$  has no section.

The following proposition is more or less implicit in the proof of ([2], Theorem 7). The approach here is different.

**Proposition 4.1.** *Let  $A$  be an abelian variety over a number field  $k$ . Fix an open subset  $U = \text{Spec}(\mathcal{O}_S)$  of the spectrum of the ring of integers in  $k$  such that  $A$  and its dual  $A^*$  extend to abelian schemes  $\mathcal{A}$  and  $\mathcal{A}^*$  over  $U$ , and let  $\ell$  be a prime number invertible on  $U$ . Assume that the Tate–Shafarevich group  $\text{III}(A)$  is finite. Then the following assertions are equivalent:*

- a) *The  $\ell$ -primary torsion subgroup  $H^1(U, \mathcal{A}^*)\{\ell\}$  is finite.*

b) *The closure of the image of the diagonal embedding*

$$A(k) \rightarrow \prod_{v|\ell} A(k_v)$$

(for the product of  $v$ -adic topologies) is a subgroup of finite index.

*Proof.* Define  $\text{III}_\ell(A^*)$  as the subgroup of  $H^1(k, A^*)$  consisting of those elements whose restriction to  $H^1(k_v, A^*)$  is zero for each  $v \nmid \ell$ . By Cassels–Tate duality ([19], Theorem I.6.13) the dual of the finite group  $\text{III}(A)$  is  $\text{III}(A^*)$ , and by Tate’s local duality ([19], Corollary I.3.4) the dual of the profinite group  $H^0(k_v, A)$  is the discrete group  $H^1(k_v, A^*)$  for each place  $v$  (here by convention for  $v$  archimedean  $H^0(k_v, A)$  means Tate’s modified cohomology group). Dualizing the exact sequence

$$0 \rightarrow \text{III}(A^*) \rightarrow \text{III}_\ell(A^*) \rightarrow \bigoplus_{v|\ell} H^1(k_v, A^*)$$

we therefore get an exact sequence

$$\prod_{v|\ell} H^0(k_v, A) \rightarrow \text{III}_\ell(A^*)^D \rightarrow \text{III}(A) \rightarrow 0$$

where the superscript  $D$  means  $\mathbf{Q}/\mathbf{Z}$ -dual.

Now by the Cassels–Tate dual exact sequence (see e.g. [10], Prop. 5.3 for a more general statement) this extends to an exact sequence

$$0 \rightarrow \overline{A(k)} \rightarrow \prod_{v|\ell} H^0(k_v, A) \rightarrow \text{III}_\ell(A^*)^D \rightarrow \text{III}(A) \rightarrow 0$$

where  $\overline{A(k)}$  denotes the closure of  $A(k)$  in the topological product. As  $\text{III}(A)$  was assumed to be finite, assertion b) is equivalent to the finiteness of  $\text{III}_\ell(A^*)$ .

On the other hand, by ([19], Lemma II.5.5) there is an exact sequence

$$(4.12) \quad 0 \rightarrow \text{III}_\ell(A^*) \rightarrow H^1(U, \mathcal{A}^*) \rightarrow \bigoplus_{v \notin U, v|\ell} H^1(k_v, A^*).$$

It follows from Fact 3.2 that for  $v$  a finite place not dividing  $\ell$  the  $\ell$ -primary torsion subgroup  $H^1(k_v, A^*)\{\ell\}$  is finite, and this holds in the archimedean case as well. Therefore by the sequence above the finiteness of  $\text{III}_\ell(A^*)\{\ell\}$  is equivalent to the finiteness of  $H^1(U, \mathcal{A}^*)\{\ell\}$ . To finish the proof, it is sufficient to show that  $\text{III}_\ell(A^*)\{\ell\}$  is of finite index in  $\text{III}_\ell(A^*)$ . There is an exact sequence

$$0 \rightarrow \text{III}(A^*) \rightarrow \text{III}_\ell(A^*) \rightarrow \bigoplus_{v|\ell} H^1(k_v, A^*)$$

The group  $\text{III}(A^*)$  is finite (this follows from the finiteness of  $\text{III}(A)$  by [19], Remark I.6.14 c)), hence  $\text{III}(A^*)/\text{III}(A^*)\{\ell\}$  is finite and  $\ell$ -divisible. On the other hand, for  $v$  dividing  $\ell$  the group  $H^1(k_v, A^*)\{\ell\}$

is of finite index in  $H^1(k_v, A^*)$  (use again Fact 3.2), and the result follows.  $\square$

**Corollary 4.2.** *If  $E$  is an elliptic curve over  $\mathbf{Q}$  with  $\text{III}(E)$  finite and  $E(\mathbf{Q})$  infinite, then  $H^1(U, \mathcal{E})\{\ell\}$  is finite, with  $U$  and  $\ell$  as above and  $\mathcal{E}$  extending  $E$  over  $U$ .*

*Proof.* Since  $E(\mathbf{Q}_\ell)$  has a finite index subgroup isomorphic to  $\mathbf{Z}_\ell$  by Fact 3.2, the assumption that  $E(\mathbf{Q})$  is infinite implies that the closure of  $E(\mathbf{Q})$  in  $E(\mathbf{Q}_\ell)$  is of finite index. Now apply the proposition.  $\square$

This is to be compared with Theorem 7 of [2] where it is proven without assuming the finiteness of  $\text{III}(E)$  that if  $E(\mathbf{Q})$  is infinite, an element of  $\text{III}(E)$  that becomes infinitely  $\ell$ -divisible in  $H^1(U, \mathcal{E})$  must be infinitely  $\ell$ -divisible in  $\text{III}(E)$ .

Based on the above corollary it is easy to construct higher dimensional examples.

**Corollary 4.3.** *Let  $A$  be an abelian variety over  $\mathbf{Q}$  that is isogenous over  $\mathbf{Q}$  to a product of elliptic curves  $E_1 \times \dots \times E_r$  such that each  $E_i$  has finite Tate-Shafarevich group and infinitely many  $\mathbf{Q}$ -points. Then  $H^1(U, \mathcal{A}^*)\{\ell\}$  and  $H^1(U, \mathcal{A})\{\ell\}$  are finite, with  $U$  and  $\ell$  as in the proposition.*

*Proof.* The finiteness of both  $\text{III}(A)$  and of  $H^1(U, \mathcal{A})\{\ell\}$  is preserved under isogeny, so we may assume  $A = E_1 \times \dots \times E_r$ . We then get product decompositions of the groups  $\text{III}(A)$ ,  $H^1(U, \mathcal{A})\{\ell\}$  and  $H^1(U, \mathcal{A}^*)\{\ell\}$  to which we may apply the previous corollary.  $\square$

**Proposition 4.4.** *Let  $A$  be an abelian variety over  $\mathbf{Q}$  satisfying the assumptions of Corollary 4.3. Then the maximal divisible subgroup of  $H^1(\mathbf{Q}, A)$  is trivial.*

*Consequently, if  $X$  is a smooth projective curve over  $\mathbf{Q}$  whose Jacobian is of the above type and which does not have a  $\mathbf{Q}$ -rational divisor class of degree 1, then the map  $\Pi^{\text{ab}}(X) \rightarrow \text{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$  has no section.*

**Remark 4.5.** The proposition is especially interesting in the case when  $X$  has a point (or at least a degree 1 divisor) defined over each completion of  $\mathbf{Q}$ , for in this case the projection  $\Pi^{\text{ab}}(X) \rightarrow \text{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$  has a section everywhere locally but not globally.

Michael Stoll has communicated to us the example of the hyperelliptic curve with affine equation  $y^2 = 3x^6 + 8x^4 + 2x^2 - 6$  whose Jacobian is the product of the rank 1 elliptic curves  $y^2 = x^3 - x^2 - 15x - 27$  and  $y^2 = x^3 - x^2 - 49x + 157$ . This Jacobian is odd in the terminology of [20], and therefore the curve has a zero-cycle of degree one everywhere locally but not globally; it thus satisfies the above assumptions.

In the appendix Victor Flynn gives explicit examples of curves of genus 2 that satisfy the above assumptions and moreover have rational points everywhere locally. This then entails that the projection  $\pi_1(X, \bar{x}) \rightarrow \text{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$ , and not just the abelianized variant, has a splitting everywhere locally but not globally. In Stoll's example there is no point over  $\mathbf{Q}_2$ .

For the proof of Proposition 4.4 we need a general lemma.

**Lemma 4.6.** *Let  $A$ ,  $\mathcal{A}$  and  $U$  be as in Proposition 4.1. An element  $\alpha \in H^1(U, \mathcal{A})$  of order invertible on  $U$  is divisible in  $H^1(U, \mathcal{A})$  if and only if its image in  $H^1(k, A)$  lies in the maximal divisible subgroup.*

*Proof.* By decomposing  $\alpha$  in its  $\ell$ -primary components we reduce to the case when  $\alpha$  has  $\ell$ -power order for a prime  $\ell$  invertible on  $U$ . By [19], Lemma II.5.5 the  $\ell$ -primary component  $H^1(U, \mathcal{A})\{\ell\}$  of the torsion group  $H^1(U, \mathcal{A})$  is of finite cotype, so its maximal divisible subgroup coincides with the subgroup of divisible elements. As the image of a divisible subgroup by a homomorphism of abelian groups is again divisible, the necessity of the condition of the proposition follows. For sufficiency recall again from ([19], Lemma II.5.5) the exact sequence

$$0 \rightarrow H^1(U, \mathcal{A}) \rightarrow H^1(k, A) \rightarrow \bigoplus_{v \in U} H^1(k_v, A).$$

Here for  $v \in U$  the group  $H^1(k_v, A)\{\ell\}$  is finite by Fact 3.2 and our assumption that  $v$  does not divide  $\ell$ . Therefore the last term of the sequence contains no nonzero  $\ell$ -divisible subgroup, so the maximal divisible subgroup of  $H^1(k, A)\{\ell\}$  is contained in  $H^1(U, \mathcal{A})$ . This is what we wanted to show.  $\square$

*Proof of Proposition 4.4.* The second assertion follows from the first via Theorem 1.2 and the fact, already noted in the introduction, that  $X$  has a  $\mathbf{Q}$ -rational divisor class of degree 1 if and only if  $\text{Alb}_X^1$  has trivial class in  $H^1(\mathbf{Q}, A)$ .

For the first assertion pick  $\alpha$  from the maximal divisible subgroup of  $H^1(\mathbf{Q}, A)$ . We may find an open subset  $U \subset \text{Spec}(\mathcal{O}_k)$  such that  $A$  extends to an abelian scheme  $\mathcal{A}$  over  $U$  and  $\alpha$  extends to an element  $\alpha_U \in H^1(U, \mathcal{A})$  of order invertible on  $U$ . By the lemma  $\alpha_U$  is divisible in  $H^1(U, \mathcal{A})$ . But by Corollary 4.3 the group  $H^1(U, \mathcal{A})\{\ell\}$  is finite for each prime  $\ell$  dividing the order of  $\alpha_U$ , so  $\alpha_U = 0$  as required.  $\square$

## 5. RELATION WITH THE ELEMENTARY OBSTRUCTION

In this section we investigate a different kind of condition for the vanishing of the class [II]. We begin with a continuous analogue for the well-known Ext spectral sequence in usual Galois cohomology.

**Proposition 5.1.** *Let  $k$  be a field,  $\Gamma = \text{Gal}(k_s|k)$ ,  $(M_n)$  a direct system of  $\Gamma$ -modules indexed by  $\mathbf{N}$ , and  $N$  a  $\Gamma$ -module. There is a spectral sequence*

$$(5.13) \quad H_{\text{cont}}^p(k, (\text{Ext}^q(M_n, N))) \Rightarrow \text{Ext}_k^{p+q}(\varinjlim M_n, N).$$

Here the notation  $\text{Ext}^q(A, B)$  stands for the  $q$ -th left derived functor of the functor  $A \mapsto \text{Hom}(A, B)$ , where  $\text{Hom}(A, B)$  consists of those homomorphisms  $A \rightarrow B$  that are invariant under some open subgroup of  $\Gamma$ . Under the correspondence between  $\Gamma$ -modules and étale sheaves on  $\text{Spec } k$  this functor corresponds to the inner Hom in the category of étale sheaves on  $\text{Spec } k$ .

*Proof.* This is the continuous analogue of the spectral sequence of [19], Example 0.8. As in the discrete case, it arises as a spectral sequence of composite functors. Let  $F$  be the functor from the category of  $\Gamma$ -modules to the category of inverse systems of  $\Gamma$ -modules given by

$$F : N \mapsto (\text{Hom}(M_n, N)),$$

and  $G$  the functor from the category of inverse systems of  $\Gamma$ -modules to that of abelian groups given by

$$G : (P_n) \mapsto (\varprojlim P_n)^\Gamma.$$

According to Jannsen's definition in [11], the  $q$ -th right derived functor of  $G$  maps  $(P_n)$  to  $H_{\text{cont}}^q(k, (P_n))$ .

We now check that  $F$  transforms injective objects to  $G$ -acyclic objects. If  $N$  is injective, then for each  $n$  the  $\Gamma$ -module  $\text{Hom}(M_n, N)$  is acyclic for the functor  $A \rightarrow A^\Gamma$ , as we see by applying Lemma 0.6 of [19] with  $M = \mathbf{Z}$ ,  $H = \{1\}$  and  $M_n$  and  $N$  in place of  $N$  and  $I$ . Therefore by [11], Proposition 1.2 the system  $(\text{Hom}(M_n, N))$  is acyclic for  $G$ .

By the previous paragraph a spectral sequence for the composite functor  $G \circ F$  exists. Its  $E_2^{pq}$ -term is  $H_{\text{cont}}^p(k, (\text{Ext}^q(M_n, N)))$ , and its abutment is the  $(p+q)$ -th right derived functor of

$$G \circ F : N \mapsto (\varprojlim \text{Hom}(M_n, N))^\Gamma.$$

But

$$\begin{aligned} (\varprojlim \text{Hom}(M_n, N))^\Gamma &\cong \varprojlim (\text{Hom}(M_n, N))^\Gamma \cong \varprojlim (\text{Hom}_{\bar{k}}(M_n, N))^\Gamma \\ &\cong (\varprojlim \text{Hom}_{\bar{k}}(M_n, N))^\Gamma \cong (\text{Hom}_{\bar{k}}(\varinjlim M_n, N))^\Gamma \cong \text{Hom}_k(\varinjlim M_n, N). \end{aligned}$$

□

**Corollary 5.2.** *Let  $k$  be a field,  $\Gamma = \text{Gal}(k_s|k)$  and  $(M_n)$  a direct system of finitely generated  $\Gamma$ -modules. For all divisible  $\Gamma$ -modules  $N$  there are isomorphisms*

$$H_{\text{cont}}^i(k, (\text{Hom}_{\bar{k}}(M_n, N))) \cong \text{Ext}_k^i(\varinjlim M_n, N).$$

*Proof.* For arbitrary  $\Gamma$ -modules  $N$  we have  $Ext^q(M_n, N) = Ext_k^q(M_n, N)$  for all  $q$  and  $n$ , because this is so for  $q = 0$  when  $M_n$  is finitely generated. If moreover  $N$  is divisible, the latter groups are trivial for  $q > 0$ .  $\square$

We now apply the above corollary in a concrete situation. *Until the end of the section we assume the base field  $k$  is of characteristic 0.*

**Corollary 5.3.** *Let  $X$  be a smooth projective geometrically integral variety over  $k$ . There exists a canonical isomorphism of abelian groups*

$$H_{\text{cont}}^2(k, T(\text{Alb}_X(\bar{k}))) \cong \text{Ext}_k^2((\text{Pic}^0 \bar{X})_{\text{tors}}, \bar{k}^\times)$$

*the extension group being taken in the category of  $\Gamma = \text{Gal}(\bar{k}|k)$ -modules.*

Here  $A_{\text{tors}}$  denotes the torsion subgroup of an abelian group  $A$ .

*Proof.* Fix a prime number  $\ell$ , and apply Corollary 5.2 with  $i = 2$ ,  $M_n = \ell^n \text{Pic}_X^0(\bar{k})$  (the  $\ell^n$ -torsion subgroup of  $\text{Pic}_X^0(\bar{k})$ ) and  $N = \bar{k}^\times$ . Then use the Weil pairing between the  $\ell^n$ -torsion of Albanese and Picard varieties to obtain

$$H_{\text{cont}}^2(k, T_\ell(\text{Alb}_X(\bar{k}))) \cong \text{Ext}_k^2((\text{Pic}^0 \bar{X})_{\ell\text{-tors}}, \bar{k}^\times).$$

The corollary follows by taking direct products over all  $\ell$ .  $\square$

Now recall (e.g. from [22], (2.16)) that there is a spectral sequence

$$E_2^{pq} = \text{Ext}_k^p((\text{Pic}^0 \bar{X})_{\text{tors}}, H^q(\bar{X}, \mathbf{G}_m)) \Rightarrow \text{Ext}_X^{p+q}(\pi^*(\text{Pic}_X^0)_{\text{tors}}, \mathbf{G}_m),$$

where  $\pi : X \rightarrow \text{Spec } k$  is the natural projection and the abutment is an extension group of étale sheaves on  $X$ . The differential  $d : E_2^{01} \rightarrow E_2^{20}$  translates as a map

$$d : \text{Hom}_\Gamma((\text{Pic}^0 \bar{X})_{\text{tors}}, \text{Pic } \bar{X}) \rightarrow \text{Ext}_k^2((\text{Pic}^0 \bar{X})_{\text{tors}}, \bar{k}^\times).$$

The natural inclusion map  $i : (\text{Pic}^0 \bar{X})_{\text{tors}} \rightarrow \text{Pic } \bar{X}$  thus yields a class  $d(i) \in \text{Ext}_k^2((\text{Pic}^0 \bar{X})_{\text{tors}}, \bar{k}^\times)$ .

**Proposition 5.4.** *Under the isomorphism of Corollary 5.3 the class [II] in  $H_{\text{cont}}^2(k, T(\text{Alb}_X(\bar{k})))$  corresponds to the class  $d(i)$  in the group  $\text{Ext}_k^2((\text{Pic}^0 \bar{X})_{\text{tors}}, \bar{k}^\times)$ .*

*Proof.* According to ([22], Theorem 2.3.4 a)), the map

$$d' : \text{Hom}_\Gamma(\text{Pic } \bar{X}, \text{Pic } \bar{X}) \rightarrow \text{Ext}_k^2(\text{Pic } \bar{X}, \bar{k}^\times)$$

coming from the spectral sequence

$$\text{Ext}_k^p(\text{Pic } \bar{X}, H^q(\bar{X}, \mathbf{G}_m)) \Rightarrow \text{Ext}_X^{p+q}(\pi^* \text{Pic } X, \mathbf{G}_m)$$

sends the identity map of  $\text{Pic } \bar{X}$  to the opposite of the class of the 2-extension

$$(5.14) \quad 1 \rightarrow \bar{k}^\times \rightarrow \bar{k}(\bar{X})^\times \rightarrow \text{Div } \bar{X} \rightarrow \text{Pic } \bar{X} \rightarrow 1.$$

The inclusion of  $\Gamma$ -modules  $(\mathrm{Pic}^0 \overline{X})_{\mathrm{tors}} \rightarrow \mathrm{Pic} \overline{X}$  gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\Gamma}(\mathrm{Pic} \overline{X}, \mathrm{Pic} \overline{X}) & \xrightarrow{d'} & \mathrm{Ext}_k^2(\mathrm{Pic} \overline{X}, \overline{k}^{\times}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\Gamma}((\mathrm{Pic}^0 \overline{X})_{\mathrm{tors}}, \mathrm{Pic} \overline{X}) & \xrightarrow{d} & \mathrm{Ext}_k^2((\mathrm{Pic}^0 \overline{X})_{\mathrm{tors}}, \overline{k}^{\times}). \end{array}$$

Therefore  $-d(i)$  is the image of the class of (5.14) by the right vertical map. The map in question factors as

$$\mathrm{Ext}_k^2(\mathrm{Pic} \overline{X}, \overline{k}^{\times}) \rightarrow \mathrm{Ext}_k^2(\mathrm{Pic}^0 \overline{X}, \overline{k}^{\times}) \xrightarrow{\lambda} \mathrm{Ext}_k^2((\mathrm{Pic}^0 \overline{X})_{\mathrm{tors}}, \overline{k}^{\times}),$$

and there is a natural map  $\Phi : \mathrm{Ext}_{k_{\acute{e}t}}^2(\mathrm{Pic}_X^0, \mathbf{G}_m) \rightarrow \mathrm{Ext}_k^2(\mathrm{Pic}^0 \overline{X}, \overline{k}^{\times})$ , where the first group is an Ext-group in the category of sheaves on the big étale site of  $k$ , and  $\Phi$  is induced by restriction of sheaves to the small étale site. The 2-extension (5.14) extends to a natural 2-extension

$$1 \rightarrow \mathbf{G}_m \rightarrow \mathcal{K}_X^{\times} \rightarrow \mathrm{Div}_X \rightarrow \mathrm{Pic}_X \rightarrow 1$$

of sheaves on the big étale site of  $k$ . Its pullback via the map  $\mathrm{Pic}_X^0 \rightarrow \mathrm{Pic}_X$  yields a class  $c \in \mathrm{Ext}_{k_{\acute{e}t}}^2(\mathrm{Pic}_X^0, \mathbf{G}_m)$  whose image by  $\lambda \circ \Phi$  is  $-d(i)$ .

Now recall that there is a natural isomorphism

$$\phi : \mathrm{Ext}_{k_{\acute{e}t}}^2(\mathrm{Pic}_X^0, \mathbf{G}_m) \xrightarrow{\sim} H^1(k, \mathrm{Alb}_X)$$

coming from the local-to-global spectral sequence

$$(5.15) \quad H^p(k, \mathrm{Ext}_{k_{\acute{e}t}}^i(\mathrm{Pic}_X^0, \mathbf{G}_m)) \Rightarrow \mathrm{Ext}_{k_{\acute{e}t}}^{p+q}(\mathrm{Pic}_X^0, \mathbf{G}_m),$$

the Barsotti-Weil isomorphism  $\mathrm{Ext}_{k_{\acute{e}t}}^1(\mathrm{Pic}_X^0, \mathbf{G}_m) \cong \mathrm{Alb}_X$  and the vanishing of  $\mathrm{Ext}_{k_{\acute{e}t}}^i(\mathrm{Pic}_X^0, \mathbf{G}_m)$  for  $i \neq 1$  (see [10], Remark 4.1 for discussion and references on this point).

Skorobogatov checked in ([23], Proposition 2.1) that  $\phi(c)$  is the opposite of the class of  $\mathrm{Alb}_X^1$ . Now consider the diagram

$$(5.16) \quad \begin{array}{ccc} \mathrm{Ext}_{k_{\acute{e}t}}^2(\mathrm{Pic}_X^0, \mathbf{G}_m) & \xrightarrow{\phi} & H^1(k, \mathrm{Alb}_X) \\ \Lambda \circ \Phi \downarrow & & \downarrow \delta \\ \mathrm{Ext}_k^2((\mathrm{Pic}^0 \overline{X})_{\mathrm{tors}}, \overline{k}^{\times}) & \longrightarrow & H_{\mathrm{cont}}^2(k, T(\mathrm{Alb}_X(\overline{k}))) \end{array}$$

where  $\delta$  is the product of the maps  $\delta_{\ell}$  introduced in (2.8). Once we have proven that the diagram commutes, it will follow from the above considerations and Proposition 2.2 that  $d(i) = -(\lambda \circ \Phi)(c)$  maps to  $[\Pi]$  in  $H_{\mathrm{cont}}^2(k, T(\mathrm{Alb}_X(\overline{k})))$ .

This commutativity is, however, a formal exercise in derived categories. As before, we shall check it by writing  $\mathrm{Ext}_k^2((\mathrm{Pic}^0 \overline{X})_{\mathrm{tors}}, \overline{k}^{\times})$  as the product of its  $\ell$ -primary components  $\mathrm{Ext}_k^2((\mathrm{Pic}^0 \overline{X})_{\ell\text{-tors}}, \overline{k}^{\times})$ ,  $T(\mathrm{Alb}_X(\overline{k}))$  as the product of the  $T_{\ell}(\mathrm{Alb}_X(\overline{k}))$ , and considering each  $\delta_{\ell}$



separately. To ease notation, we set  $M = \text{Pic}_X^0$ ,  $N = \mathbf{G}_m$ ,  $M_n = {}_{\ell^n}\text{Pic}_X^0$ , these being sheaves on the big étale site  $k_{\text{ét}}$ .

The upper horizontal map  $\phi$  then comes from the isomorphism

$$\mathbf{R}\text{Hom}_{k_{\text{ét}}}(M, N) \cong \mathbf{R}\Gamma(k, \mathbf{R}\text{Hom}_{k_{\text{ét}}}(M, N)),$$

which is the derived category version of spectral sequence (5.15). Similarly, the lower horizontal map comes from

$$\mathbf{R}\text{Hom}_k(\varinjlim M_n, N) \cong \mathbf{R}(\varprojlim \mathbf{R}\text{Hom}_k(M_n, N))^\Gamma,$$

the derived category version of spectral sequence (5.13). Here the  $\text{Hom}$ - and  $\text{Hom}$ -functors concern the restrictions of sheaves to the small étale site of  $k$ . The natural map  $\varinjlim M_n \rightarrow M$  induces a commutative diagram

$$\begin{array}{ccc} \mathbf{R}\text{Hom}_{k_{\text{ét}}}(M, N) \cong \mathbf{R}\Gamma(k, \mathbf{R}\text{Hom}_{k_{\text{ét}}}(M, N)) & & \\ \downarrow & & \downarrow \\ \mathbf{R}\text{Hom}_k(\varinjlim M_n, N) \cong \mathbf{R}(\varprojlim \mathbf{R}\text{Hom}_k(M_n, N))^\Gamma & & \end{array}$$

We claim that the  $\ell$ -primary part of diagram (5.16) arises from the above by applying the functor  $H^2$ . Indeed, we have already recalled above the vanishing of  $\text{Ext}_{k_{\text{ét}}}^i(M, N)$  for  $i \neq 1$ , so

$$H^j(k, \mathbf{R}\text{Hom}_{k_{\text{ét}}}(M, N)) \cong H^{j-1}(k, \text{Ext}_{k_{\text{ét}}}^1(M, N))$$

for all  $j > 0$ . The Barsotti–Weil formula then identifies the right-hand-side group for  $j = 2$  with the upper right group in (5.16). Moreover, we have seen in the proof of Corollary 5.2 that the groups  $\text{Ext}^i(M_n, N)$  vanish for  $i > 0$ , whence isomorphisms

$$H_{\text{cont}}^j(k, (\mathbf{R}\text{Hom}_k(M_n, N))) \cong H_{\text{cont}}^j(k, (\text{Hom}_k(M_n, N)))$$

and we obtain the lower right group in (5.16) for  $j = 2$ . The only point that still needs some justification is the identification of the map

$$H^1(k, \text{Ext}_{k_{\text{ét}}}^1(\text{Pic}_X^0, \mathbf{G}_m)) \rightarrow H_{\text{cont}}^2(k, (\text{Hom}_k({}_{\ell^n}\text{Pic}_X^0, \bar{k}^\times)))$$

with  $\delta_\ell$ . This is because the Barsotti–Weil formula translates the exact sequence

$$0 \rightarrow {}_{\ell^n}\text{Alb}_X \rightarrow \text{Alb}_X \xrightarrow{\ell^n} \text{Alb}_X \rightarrow 0$$

of sheaves on  $k_{\text{ét}}$  to

$$0 \rightarrow \text{Hom}_{k_{\text{ét}}}({}_{\ell^n}\text{Pic}_X^0, \mathbf{G}_m) \rightarrow \text{Ext}_{k_{\text{ét}}}^1(\text{Pic}_X^0, \mathbf{G}_m) \xrightarrow{\ell^n} \text{Ext}_{k_{\text{ét}}}^1(\text{Pic}_X^0, \mathbf{G}_m) \rightarrow 0$$

in view of  $\text{Hom}_{k_{\text{ét}}}(\text{Pic}_X^0, \mathbf{G}_m) = 0$ .  $\square$

According to ([22], Theorem 2.3.4 b)) the class (5.14) is trivial if and only if the exact sequence

$$(5.17) \quad 1 \rightarrow \bar{k}^\times \rightarrow \bar{k}(\bar{X})^\times \rightarrow \bar{k}(\bar{X})^\times / \bar{k}^\times \rightarrow 1$$

has a Galois-equivariant section. The latter property is called, following Colliot-Thélène and Sansuc, the vanishing of the *elementary obstruction*. Therefore Proposition 5.4 implies:

**Corollary 5.5.** *If the elementary obstruction vanishes, the projection  $\Pi \rightarrow \text{Gal}(\bar{k}|k)$  has a section.*

Note that the converse is not true (there are plenty of rational, hence geometrically simply connected varieties with nontrivial elementary obstruction).

**Remark 5.6.** An argument similar to the one proving Proposition 5.3 gives an isomorphism

$$H_{\text{cont}}^2(k, \Pi^{\text{ab}}(X)) \cong \text{Ext}_k^2((\text{Pic } \bar{X})_{\text{tors}}, \bar{k}^\times).$$

Whence reasoning as above we obtain that the vanishing of the elementary obstruction implies the existence of a section for the projection  $\Pi^{\text{ab}}(X) \rightarrow \text{Gal}(\bar{k}|k)$  as well. Hélène Esnault and Olivier Wittenberg tell us that they have independently obtained a similar result.

## 6. APPENDIX: AN EXAMPLE SATISFYING THE CONDITIONS OF PROPOSITION 1.3

*by E. V. Flynn*

In this appendix, we shall show the existence of an example which satisfies the conditions of Proposition 1.3, that is, a smooth projective curve over  $\mathbb{Q}$ , with points everywhere locally, with no  $\mathbb{Q}$ -rational divisor class of degree 1, and whose Jacobian is isogenous over  $\mathbb{Q}$  to a product of elliptic curves (each defined over  $\mathbb{Q}$ ), each of which has finite Tate-Shafarevich group and infinitely many  $\mathbb{Q}$ -rational points. We first recall the following theorem of Kolyvagin [14].

**Theorem 6.1.** *Let  $\mathcal{E}$  be an elliptic curve, defined over  $\mathbb{Q}$ . If  $\mathcal{E}(\mathbb{Q})$  has analytic rank 0 or 1, then  $\text{III}(\mathcal{E}/\mathbb{Q})$  is finite and the rank of  $\mathcal{E}(\mathbb{Q})$  is equal to its analytic rank.*

The above result was originally proved in [14], subject to the condition that  $\mathcal{E}$  is modular; since we now know [4] that all elliptic curves over  $\mathbb{Q}$  are modular, the result can be given, as above, without needing to state the modularity condition. It is then natural to seek an example for which the Jacobian is isogenous over  $\mathbb{Q}$  to a product of elliptic curves, each of rank 1. There are various examples in the literature (for example, [8]) of curves violating the Hasse principle, which make use of the existence or non-existence of a rational divisor class of degree 1; however, none of these examples satisfy the conditions of Proposition 1.3. It seems likely that some of the curves in [5] should satisfy these conditions, however it is difficult to identify from the published data which specific curves do so.

We shall seek a genus 2 example

$$(6.18) \quad \mathcal{C} : Y^2 = F(X) = f_6X^6 + f_5X^5 + f_4X^4 + f_3X^3 + f_2X^2 + f_1X + f_0,$$

where  $f_0, \dots, f_6 \in \mathbb{Q}$  and  $f_6 \neq 0$ . Let  $J$  denote the Jacobian of  $\mathcal{C}$ . Following Chapter 1 of [6], the model (6.18) should be taken as shorthand for the corresponding projective smooth curve, and we let  $\infty^+, \infty^-$  denote the points on the non-singular curve that lie over the singular point on (6.18) at infinity. For any field  $k$ , with  $\mathbb{Q} \subseteq k$ , we have  $\infty^+, \infty^- \in \mathcal{C}(k)$  when the coefficient of  $X^6$  is a square in  $k$ . We shall adopt the customary shorthand notation  $\{P_1, P_2\}$  to denote the divisor class  $[P_1 + P_2 - \infty^+ - \infty^-]$ , which is in  $J(\mathbb{Q})$  when  $P_1, P_2$  are points on  $\mathcal{C}$  and either  $P_1, P_2$  are both  $\mathbb{Q}$ -rational or  $P_1, P_2$  are quadratic over  $\mathbb{Q}$  and conjugate. Let  $F(X) = F_1(X) \dots F_m(X)$  be the factorisation of  $F(X)$  into irreducible polynomials over  $\mathbb{Q}$ ; for each  $i$ , let  $\theta_i$  be a root of  $F_i(X)$  and let  $L_i = \mathbb{Q}(\theta_i)$ . Following p.49 of [6], we define the homomorphism

$$(6.19) \quad \begin{aligned} \mu : J(\mathbb{Q}) &\rightarrow \left( L_1^*/(L_1^*)^2 \times \dots \times L_m^*/(L_m^*)^2 \right) / \sim, \\ \{(x_1, y_1), (x_2, y_2)\} &\mapsto [(x_1 - \theta_1)(x_2 - \theta_1), \dots, (x_1 - \theta_m)(x_2 - \theta_m)], \end{aligned}$$

where the equivalence relation  $\sim$  is defined by

$$[a_1, \dots, a_m] \sim [b_1, \dots, b_m] \Leftrightarrow a_1 = wb_1, \dots, a_m = wb_m, \text{ for some } w \in \mathbb{Q}^*.$$

Since  $\mu$  is a map to a Boolean group, its kernel clearly contains  $2J(\mathbb{Q})$ . The following result (Lemma 6 in [8]) gives a way of showing the non-existence of a  $\mathbb{Q}$ -rational divisor class of degree 1.

**Lemma 6.2.** *Let  $\mathcal{C} : Y^2 = F(X)$  be as in (6.18) defined over  $\mathbb{Q}$ , with Jacobian  $J$ , and suppose that*

- (i)  $F(X)$  has no root  $\theta \in \mathbb{Q}$ .
- (ii) *The roots of  $F(X)$  cannot be divided into two sets of three roots, where the sets are either defined over  $\mathbb{Q}$  (as wholes) or defined over a quadratic extension and conjugate over  $\mathbb{Q}$ .*

*If the kernel of  $\mu$  is  $2J(\mathbb{Q})$  then there does not exist a  $\mathbb{Q}$ -rational divisor class of degree 1 on  $\mathcal{C}$ . In particular,  $\mathcal{C}(\mathbb{Q}) = \emptyset$ .*

We are now in a position to give our example. We take a prime  $p \equiv 7 \pmod{8}$ , as well as  $a \in \mathbb{Z}, a \neq p, 2p$ , and let  $\mathcal{C}_{p,a}$  denote

$$(6.20) \quad \mathcal{C}_{p,a} : Y^2 = 2(X^2 + p)(X^2 + 2p)(X^2 + a).$$

**Example 6.3.** *The curve  $\mathcal{C}_{7,-11} : Y^2 = 2(X^2 + 7)(X^2 + 14)(X^2 - 11)$  has points everywhere locally, but has no  $\mathbb{Q}$ -rational divisor class of degree 1 and so  $\mathcal{C}_{7,-11}(\mathbb{Q}) = \emptyset$ ; the Jacobian  $J_{7,-11}$  is isogenous over  $\mathbb{Q}$  to two elliptic curves defined over  $\mathbb{Q}$ , each of rank 1, and each with finite Tate-Shafarevich group. Hence  $\mathcal{C}_{7,-11}$  satisfies the conditions of Proposition 1.3.*

*Proof.* First note that, for all  $\mathbb{Q}_p$  at least one of  $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}_p$  exist such that  $\gamma_1^2 = 2, \gamma_2^2 = -7$  or  $\gamma_3^2 = -14$ , and so at least one of  $\infty^+, (\gamma_2, 0)$  or  $(\gamma_3, 0) \in \mathcal{C}_{7,-11}(\mathbb{Q}_p)$ ; a similar statement holds over  $\mathbb{R}$ . Hence  $\mathcal{C}_{7,-11}$  has points everywhere locally. Furthermore, there are maps  $(x, y) \mapsto (x^2, y)$  and  $(x, y) \mapsto (1/x^2, y/x^3)$  from  $\mathcal{C}_{7,-11}$  to the elliptic curves

$$Y^2 = 2(X + 7)(X + 14)(X - 11)$$

and

$$Y^2 = 2(1 + 7X)(1 + 14X)(1 - 11X),$$

and  $J_{7,-11}$  is isogenous over  $\mathbb{Q}$  to their product (a special case of the isogeny in Theorem 14.1.1 of [6]).

Furthermore, each of these elliptic curves can be shown to have analytic rank 1 (for example, using Magma [16]) and so by Theorem 6.1 they each have rank 1 and finite Tate-Shafarevich group. Since their product is isogenous over  $\mathbb{Q}$  to  $J_{7,-11}$ , it also follows that the rank of  $J_{7,-11}(\mathbb{Q})$  is 2. Clearly  $J_{7,-11}(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , since points of order 2 in  $J_{7,-11}(\mathbb{Q})$  correspond to quadratics  $q(X)$ , defined over  $\mathbb{Q}$ , with  $q(X) \mid 2(1 + 7X)(1 + 14X)(1 - 11X)$  (see p. 3 of [6]) and there are three such  $q(X)$ . So (since the rank is 2), we have that  $J_{7,-11}(\mathbb{Q})/2J_{7,-11}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . It only remains to show that  $\mathcal{C}_{7,-11}$  has no rational divisor class of degree 1. After a short search on  $J_{7,-11}(\mathbb{Q})$  one finds

$$T_1 = \{(\sqrt{-7}, 0), (-\sqrt{-7}, 0)\}, T_2 = \{(\sqrt{-14}, 0), (-\sqrt{-14}, 0)\},$$

$$D_1 = \{(\sqrt{-\frac{23}{2}}, \frac{45}{2}\sqrt{-\frac{23}{2}}), (-\sqrt{-\frac{23}{2}}, -\frac{45}{2}\sqrt{-\frac{23}{2}})\},$$

$$D_2 = \{(\sqrt{-\frac{77}{23}}, \frac{420}{23}\sqrt{-\frac{77}{23}}), (-\sqrt{-\frac{77}{23}}, -\frac{420}{23}\sqrt{-\frac{77}{23}})\}.$$

Applying the map  $\mu$  of (6.19) to  $n_1T_1 + n_2T_2 + n_3D_1 + n_4D_2$ , for all 16 choices of  $n_i = 0, 1$ , we find that only the case  $n_1 = n_2 = n_3 = n_4 = 0$  is mapped by  $\mu$  to the identity and so  $T_1, T_2, D_1, D_2$  are independent in  $J_{7,-11}(\mathbb{Q})/(\ker \mu)$ . Since  $2J_{7,-11}(\mathbb{Q}) \subseteq \ker \mu$  these must also be independent in  $J_{7,-11}(\mathbb{Q})/2J_{7,-11}(\mathbb{Q})$ ; since also  $J_{7,-11}(\mathbb{Q})/2J_{7,-11}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  it follows that  $T_1, T_2, D_1, D_2$  generate  $J_{7,-11}(\mathbb{Q})/2J_{7,-11}(\mathbb{Q})$  and that  $\ker \mu = 2J_{7,-11}(\mathbb{Q})$ . The roots of  $2(X^2 + 7)(X^2 + 14)(X^2 - 11)$  clearly satisfy (i), (ii) of Lemma 6.2, from which we deduce that there does not exist a  $\mathbb{Q}$ -rational divisor class of degree 1 on  $\mathcal{C}_{7,-11}$ , as required.  $\square$

All curves in the family (6.20) have points everywhere locally, by the same argument as given for Example 6.3. The above example was found by initially searching in Magma [16] the family (6.20), with  $p \leq 50$  and  $-20 \leq a \leq 20$ , for cases where  $J_{p,a}$  is isogenous over  $\mathbb{Q}$  to the product

of two rank 1 elliptic curves, and  $\ker \mu = 2J_{p,a}(\mathbb{Q})$ . There were eight such pairs  $(p, a)$ :

$(7, -19), (7, -11), (23, 13), (31, -14), (31, -11), (31, 5), (31, 13), (47, 13)$ .

For all of these pairs,  $\mathcal{C}_{p,a}$  has no  $\mathbb{Q}$ -rational divisor class of degree 1, by the same argument as given for Example 6.3, and so all of these satisfy the conditions of Proposition 1.3.

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