Honour Moderations: Linear Algebra
Problem Sheet 1
(To be done in Third Week)

Michaelmas Term 2005

1. (a) Find a vector perpendicular to all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x + y + z = 0$.

(b) Find two vectors $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, neither of which is a scalar multiple of the other, such that the co-ordinates of both satisfy $x + y + z = 0$.

(c) Show that a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies $x+y+z = 0$ if and only if there exist scalars c_1 and c_2 such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}.$$

What is the geometrical significance of this?

(d) Is the following statement true or false?
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, so the planes $x+y+z = 0$ and $x+y-2z = 0$ are orthogonal.

2. Prove that if I, J and K are the complex matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

respectively, then $I^2 = J^2 = K^2 = -1$, $IJ = -JI = K$, $JK = -KJ = I$, and $KI = -IK = J$.

3. For each of the following values of the 2×2 matrix A, evaluate the product $A \begin{pmatrix} x \\ y \end{pmatrix}$, and give a geometric interpretation of the function taking $\begin{pmatrix} x \\ y \end{pmatrix}$ to $A \begin{pmatrix} x \\ y \end{pmatrix}$.

(i) $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ $(k \in \mathbb{R})$
(remember to take into account the sign of k when giving your interpretation);

(ii) $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ $(k \in \mathbb{R})$;

(iii) $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $(\theta \in \mathbb{R})$
(**hint:** do the much easier special case $\theta = 0$ first).

4. (a) For $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find A^{-1}, B^{-1} and $(AB)^{-1}$.

(b) Show that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse if and only if $ad - bc \neq 0$.
Find the inverse of A.

(c) Determine all 2×2 matrices A with real entries such that $A^2 = I$.

5. (i) Let A and B be $n \times n$ matrices with A symmetric and B skew-symmetric. Determine which of the following are symmetric and which are skew-symmetric:

(a) $AB + BA$;

(b) $AB - BA$;

(c) A^2;

(d) B^2.

(ii) Let A be an $n \times n$ matrix over $\mathbb{R}$, so $A = (a_{ij}), 1 \leq i \leq n, 1 \leq j \leq n$, where $a_{ij} \in \mathbb{R}$. We define the trace of A as follows:

$$\text{trace}A = \sum_{i=1}^{n} a_{ii} = \text{tr}A.$$

Show that if B is another $n \times n$ matrix over $\mathbb{R}$, then $\text{tr}(AB) = \text{tr}(BA)$.

G.A.S.