# ANALYSIS I

# 1 The real number system

## 1.1 What are the reals?

What are the *real numbers*? For the moment this is too hard a question!
We can give various answers, but for the moment we prefer just to agree a set of *axioms*—statements we will assume are true about real numbers. We will base all our arguments on these *axioms* and on nothing else, and develop all the mathematics we've learned before from these axioms alone.

## 1.2 Axioms

Naively, then we assume we've got a set $\mathbb{R}$ which we call *the real numbers* which satisfies the axioms we're going to list.

## 1.3 Addition

(i) For every ordered pair of real numbers $a$, $b$ we can associate a third one written $a+b$ called their **sum**

(ii) To every real number $a$ we can associate an other one written $-a$ called its **negative**

(iii) There is a special real number 0 called **zero**

such that

$$a + b = b + a \tag{A1}$$
$$a + (b + c) = (a + b) + c \tag{A2}$$
$$a + 0 = a \tag{A3}$$
$$a + (-a) = 0 \tag{A4}$$

## 1.4 Easy properties of A1—A4

**1.4.1.** *If* $a + x = 0$ *and* $a + y = 0$ *then* $x = y$

*Proof.*

$$\begin{array}{rcll} y & = & y + 0 & \text{(A3)} \\ & = & y + (a + x) & \textit{Assumption} \\ & = & (y + a) + x & \text{(A2)} \\ & = & (a + y) + x & \text{(A1)} \\ & = & 0 + x & \textit{Assumption} \\ & = & x + 0 & \text{(A1)} \\ & = & x & \text{(A3)} \end{array}$$

□

**1.4.2.** $-(-a) = a$

*Proof.*

$$\begin{aligned} (-a)+a &= a+(-a) && \text{(A1)} \\ &= 0 && \text{(A4)} \\ (-a)+(-(-a)) &= 0 && \text{(A4)} \\ a &= (-(-a)) && \text{by (1.4.1)} \end{aligned}$$

□

**1.4.3.** $-(a+b) = (-a)+(-b)$

*Proof.*

$$\begin{aligned} (a+b)+((-a)+(-b)) &= ((a+b)+(-a))+(-b) && \text{(A2)} \\ &= ((b+a)+(-a))+(-b) && \text{(A1)} \\ &= (b+(a+(-a)))+(-b) && \text{(A2)} \\ &= (b+0)+(-b) && \text{(A4)} \\ &= b+(-b) && \text{(A3)} \\ &= 0 && \text{(A4)} \end{aligned}$$

Also $(a+b)+(-(a+b)) = 0$ by (A4). So by (1.4.1)

$$-(a+b) = (-a)+(-b).$$

□

**1.4.4.** $-0 = 0$

*Proof.*

$$\begin{aligned} 0+0 &= 0 && \text{(A3)} \\ 0+(-0) &= 0 && \text{(A4)} \end{aligned}$$

therefore $0 = -0$ by (1.4.1). □

## 1.5 Multiplication

(i) To every ordered pairs of real numbers $a$, $b$ we can associate a third one written $a \cdot b$ called their **product**

(ii) To every real number except 0 we can associate an other one written $1/a$ called its reciprocal

(iii) There is a special real number 1

such that

$$a \cdot b = b \cdot a \tag{M1}$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{M2}$$

$$a \cdot 1 = a \tag{M3}$$

$$a \cdot \frac{1}{a} = 1 \text{ if } a \neq 0 \tag{M4}$$

## 1.6 Easy Consequences of M1—M4

**1.6.1.** *If* $a \neq 0$ *and* $a \cdot x/a \cdot y$ *then* $x = y$.

**1.6.2.** *If* $a \neq 0$ *then* $1/(1/a)) = a$.

**1.6.3.** *If* $a \neq 0$ *and* $b \neq 0$ *then*

Note that as M1—M4 say the same things about $\cdot$ as A1—A4 say about $+$ we can just translate the proofs.

## 1.7 The Distributive Law

For all $a$, $b$, $c$ numbers

$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{D}$$

## 1.8 More Consequences

**1.8.1.** $(a + b) \cdot c = a \cdot c + b \cdot c$

*Proof.*

$$\begin{aligned} (a+b) \cdot c &= c \cdot (a+b) && \text{(M1)} \\ &= c \cdot a + c \cdot b && \text{(D)} \\ &= a \cdot c + b \cdot c && \text{(M1)}^2 \end{aligned}$$

□

**1.8.2.** $a \cdot 0 = 0$

*Proof.*

$$a \overset{\text{(M3)}}{=} a \cdot 1 \overset{\text{(A3)}}{=} a \cdot (1 + 0) \overset{\text{(D)}}{=} a \cdot 1 + a \cdot 0 \overset{\text{(M3)}}{=} a + (a \cdot 0)$$

Therefore $a = (a \cdot 0) + a$. Now

$$a + (-a) = ((a \cdot 0) + a) + (-a)$$

and so $0 = a$ by an earlier result. □

**1.8.3.** $a \cdot (-b) = -(a \cdot b)$

*Proof.*

$$\begin{aligned} (a \cdot b) + (a \cdot (-b)) &= a \cdot (b + (-b)) && \text{(D)} \\ &= a \cdot 0 && \text{(A4)} \\ &= 0 \\ (a \cdot b) + (-(a \cdot b)) &= 0 \end{aligned}$$

So by (1.4.1)

$$a \cdot (-b) = -(a \cdot b).$$

□

**1.8.4.** $(-1) \cdot (-1) = 1$

*Proof.*

$$\begin{aligned} (-1) \cdot (-1) &= (-(-1) \cdot )1)) \\ &= (-(-1)) && \text{(M3)} \\ &= 1 && \text{by (1.4.1)} \end{aligned}$$

□

## 1.9 Avoiding Collapse

If $1 = 0$ we would have

$$x = x \cdot 1 = x \cdot 0 = 0 \text{ for all } x$$

We must assume

$$0 \neq 1 \tag{Z}$$

**Note.** *This is the only "safe" contradiction in mathematics.*

## 1.10 Notation

We write

$$\begin{array}{ccc} ab & \text{for} & a \cdot b \\ a - b & \text{for} & a + (-b) \\ a/b & \text{for} & a \cdot (1/b) \\ a^{-1} & \text{for} & 1/a \end{array}$$

Also we write

$$\begin{array}{rcll} a^0 & = & 1 & \\ a^{k+1} & = & a^k \cdot a & \text{for all } k \in \mathbb{N} \\ a^{-l} & = & 1/a^l & \text{for all } l \in \mathbb{N} \end{array}$$

## 1.11 Other systems

**Note.** *Other systems also satisfy* (A1)-(A4)*,* (M1)-(M4),(D)*,* (Z)*. They are called* fields*.*

**Example.** $\mathbb{Q}$*,* $\mathbb{R}$*,* $\mathbb{C}$*, but more exciting ones exist.*

**Secret.** All that you do in Linear Algebra uses only these axioms.
**Fact.** All the arithmetic properties of $\mathbb{R}$ can be easily deduced from those we have done. We cease to labour this point.

# 2 Putting order on $\mathbb{R}$

## 2.1 The number 'line'

We want to capture the idea that the real numbers are 'ordered' in an axiom. It is easier just to give axioms for being positive.
There is a subset $\mathbb{P}$ of $\mathbb{R}$ called the **positive** real numbers satisfying:

$$a, b \in \mathbb{P} \longrightarrow a + b \in \mathbb{P} \tag{P1}$$
$$a, b \in \mathbb{P} \longrightarrow ab \in \mathbb{P} \tag{P2}$$

$$\text{Exactly one of } \ a \in \mathbb{P}, \ a = 0, \ -a \in \mathbb{P} \tag{P3}$$

## 2.2 Easy consequences

**2.2.1.** $1 \in \mathbb{P}$

*Proof.* By (P3) one of one of

$$\begin{array}{rcl} 1 \in \mathbb{P} & \Rightarrow & n \in \mathbb{P} \\ 1 = 0 & & \text{contradicts to } \ (Z) \\ -1 \in \mathbb{P} & ? & \text{Well, let's see} \ldots \end{array}$$

If so, by (P1), $(-1)(-1) \in \mathbb{P}$. By (1.8.4), $1 \in \mathbb{P}$. Now $1 \in \mathbb{P}$ and $(-1) \in \mathbb{P}$ which contradicts to (P3). □

**2.2.2.** *For all $x$, $(x + 1) - x \in \mathbb{P}$.*

*Proof.* By the arithmetic axioms, $LHS = 1$. □

## 2.3 Notation

We write

$$\begin{array}{lll} a > b & \text{for} & a - b \in \mathbb{P} \\ a < b & \text{for} & b - a \in \mathbb{P} \\ a \geqslant b & \text{for} & a - b \in \mathbb{P} \text{ or } a - b = 0 \\ a \leqslant b & \text{for} & b - a \in \mathbb{P} \text{ or } b - a = 0 \end{array}$$

## 2.4 Easy consequences

For all $x$, $y$, $z$

**2.4.1.** $x \leqslant x$

**2.4.2.** $x \leqslant y$ *and* $y \leqslant x \implies x = y$

**2.4.3.** $x \leqslant y$ *and* $y \leqslant z \implies x \leqslant z$

*Proof.*

(2.4.1) By (A4) $x - x = 0$ so $x \leqslant x$

(2.4.2) Given

$$\begin{aligned} x - y \in \mathbb{P} &\text{ or } x - y = 0 \\ y - x \in \mathbb{P} &\text{ or } y - x = 0 \end{aligned}$$

Note, $-(x-y) = -x + - - y = -x + y = y - x$.

So by (P3) not $x - y \in \mathbb{P}$ and $y - x \in \mathbb{P}$. So either $x - y = 0$ or $y - x = 0$ so $x = y$.

(2.4.3) Given

$$\begin{aligned} y - x \in \mathbb{P} &\text{ or } y - x = 0 \\ z - y \in \mathbb{P} &\text{ or } z - y = 0 \end{aligned}$$

Cases

$$\begin{aligned} y - x \in \mathbb{P} \text{ and } z - y \in \mathbb{P} &\implies (y-x) + (z-y) \in \mathbb{P} \text{ by (P1), i.e. } z - x \in \mathbb{P} \surd\surd \\ y - x \in \mathbb{P} \text{ and } z - y = 0 &\implies (y-x) + (z-y) = (y-x) \in \mathbb{P} \surd\surd \\ y - x = 0 \text{ and } z - y \in \mathbb{P} &\implies (y-x) + (z-y) = (z-y) \in \mathbb{P} \surd\surd \\ y - x = 0 \text{ and } z - y = 0 &\implies (y-x) + (z-y) = 0 + 0 = 0 \surd\surd \end{aligned}$$

□

## 2.5 Inequalities shift

**2.5.1.** $x > y \implies x + z > y + z$

*Proof.* $(x+z) - (y+z) = x - y$. □

## 2.6 Two important functions: max, min

Define $\max : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\max(x, y) = \begin{cases} x & \text{if } x \geqslant y \\ y & \text{if } y > x \end{cases}$$

By trichotomy $\implies$ well defined function.
And then min by

$$\min(x, y) = \begin{cases} y & \text{if } x \geqslant y \\ x & \text{if } y > x \end{cases}$$

**Note.** *We can extend it to a function of many variables:*

$$\max(a_1, a_2, \ldots, a_{n+1}) = \max(\max(a_1, \ldots, a_n), a_{n+1})$$

*but we will let you do that yourself.*

**Exercise.** $\max(x, y) = -\min(-x, -y)$, *because*

| | $\max(x,y)$ | | $-\min(-x,-y)$ |
|---|---|---|---|
| $x > y$ | $x$ | $-x < -y$ | $- - x = x$ |
| $x = y$ | $x$ | $x = y$ | |
| $x < y$ | $y$ | $-y < -x$ | |

## 2.7 An important function—Modulus

We define $|\ |: \mathbb{R} \to \mathbb{R}$ by

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

These cases are distinct and cover all possibilities by (P3), so we get a well-defined function.

## 2.8 An easy consequence

**2.8.1.** $|-x| = |x|$

*Proof.*

- If $x > 0$ then $x \neq 0$ so $-x \neq 0$. By (P3) then $(-x) < 0$. So $|-x| = -(-x) = x = |x|$.
- If $x = 0$ then $-x = -0 = 0$ (1.4.4). So $|-x| = 0 = |x|$.
- If $x < 0$ then $-x > 0$. So $|-x| = -x = |x|$.

□

## 2.9 The Triangle Law ($\Delta$ )

For all $a$, $b$

$$|a + b| \leqslant |a| + |b|$$

*Proof.* There are possible 8 cases

| | $a$ | $b$ | $a+b$ | |
|---|---|---|---|---|
| $(A)$ | $\geqslant 0$ | $\geqslant 0$ | $\geqslant 0$ | |
| | $\geqslant 0$ | $\geqslant 0$ | $< 0$ | Forbidden by (P1) |
| $(B)$ | $\geqslant 0$ | $< 0$ | $\geqslant 0$ | |
| $(C)$ | $\geqslant 0$ | $< 0$ | $< 0$ | |
| $(B')$ | $< 0$ | $\geqslant 0$ | $\geqslant 0$ | |
| $(C')$ | $< 0$ | $\geqslant 0$ | $< 0$ | |
| | $< 0$ | $< 0$ | $\geqslant 0$ | Forbidden by (P1) |
| $(D)$ | $< 0$ | $< 0$ | $< 0$ | |

(A) $|a| = a$, $|b| = b$, $|a + b| = a + b$

(D) $|a| = -a$, $|b| = -b$, $|a + b| = -(a + b)$

(B) $|a| = a$, $|b| = -b$, $|a + b| = a + b$
Is $(a - b) \geqslant (a + b)$? ie Is $(-b) + (-b) \geqslant 0$? Yes, as $(-b) \in \mathbb{P}$ by (P1).

(C) $|a| = a$, $|b| = -b$, $|a + b| = -(a + b)$
Is $-(a + b) \leqslant a - b$? I.e. is $0 \leqslant a + a$? Yes, $a \in \mathbb{P}$ by (P1).

(B') , (C') got by swapping $a$ and $b$

□

## 2.10 The modulus of a product

$|ab| = |a||b|$

*Proof.* For you to do; see Exercise Sheets. □

## 2.11 Arguing with Inequalities

The rules we use in practice are

- 2.4.1, 2.4.2, 2.4.3;
- $|ab| = |a||b|$;
- the $\Delta$ Law.