ANALYSIS I

3 Completing our description of $\mathbb{R}$

3.1 Recap

At this stage we can surely persuade ourselves that we could write down proofs of all the usual *algebraic* properties of $\mathbb{R}$, and all the usual properties of "$\leqslant$".
But note, many structures share these properties; in particular $\mathbb{Q}$ and $\mathbb{R}$.

3.2 Gaps in $\mathbb{Q}$

Why won't $\mathbb{Q}$ do? The Greeks told us—because in $\mathbb{Q}$ we can't find an element to measure the length of the hypotenuse of a right angled isosceles triangle with two short sides of length 1.

Theorem. *There is no element $\alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$.*

Proof. If there were, $\alpha = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Chose these in lowest terms. Then $2n^2 = m^2$. So LHS is also even. As odd×odd=odd we get that m is even. So $n^2 = 2m_0^2$ & so n is even.#. □

Note. *These properties of $\mathbb{N}$ we can prove by induction... a bit of challenge for you.*

3.3 Greatest and Least Elements

Let $B \in \mathbb{R}$. We say that

- b_1 is **least element** of B if $\begin{array}{ll} (i) & b_1 \in B \\ (ii) & b \in B \Rightarrow b \geqslant b_1 \end{array}$

- b_2 is **greatest element** of B if $\begin{array}{ll} (i) & b_2 \in B \\ (ii) & b \in B \Rightarrow b \leqslant b_2 \end{array}$

E.g. 1 is least element of $[1, 2)$. Greatest element? Hmm?!!

3.4 Uniqueness of these

3.4.1. *A least(greatest) element is unique.*

Proof. Suppose that b_1 and b_2 are both greatest elements of B. Then as $b_1 \in B$ and b_2 is greatest, $b_1 \leqslant b_2$; as $b_2 \in B$ and b_1 is greatest, $b_2 \leqslant b_1$. Hence $b_1 = b_2$. (Trichotomy). □

3.5 Upper and Lower Bounds

Let $B \in \mathbb{R}$. We say that

- h_1 is a **lower bound** of B if $b \in B \Rightarrow b \geqslant h_1$.

- h_2 is a **upper bound** of B if $b \in B \Rightarrow b \leqslant h_2$

Example. 23 *is upper bound of* $[1, 2)$*, so is* $16\frac{3}{4}$
1 *is a lower bound of* $[1, 2)$*. So is* -37.

3.6 The Completeness Axiom

Axiom C: let $E \neq \emptyset$ be bounded above. Then the set of upper bounds has a least element. We write this *unique* element $\sup E$, and call it the supremum of E.

3.6.1. *2 is supremum of* $[1, 2)$

Proof. For all $x \in [1, 2)$, $1 \leqslant x < 2$ by definition, so clearly 2 is an upper bound. Now suppose that there was a smaller upper bound, t. So $t < 2$, and as it is an upper bound, $t \geqslant 1$. Then

$$\frac{3}{2} \leqslant \frac{t+2}{2} < 2$$

(Follows from $3 \leqslant t + 2 < 4$.) So

$$\frac{t+2}{2} \in [1, 2)$$

and so

$$\frac{t+2}{2} \leqslant t \Rightarrow t \geqslant 2$$

which is contradiction. □

3.7 The infimum

Theorem. *Let $F \neq \emptyset$ be a set which is bounded below. Then the set of lower bounds of F has a greatest element.*

Note. *It is unique and we denote it* $\inf F$*, the infimum of F.*

Proof. The proof is done by standing on our heads and using the Completeness Axiom.
Let $E = \{-x | x \in F\}$. (Observe taht $x \leqslant y \iff -y \leqslant -x$) Then $E \neq \emptyset$. Let l be a lower bound of F. Then $l \leqslant x$ for $\forall x \in F$. So $-x \leqslant -l$ for all $x \in F$. As $y \leqslant -l$ for $\forall y \in E$. So E is bounded above. By Completeness Axiom, $\sup E$ exists.
We prove that

(i) $-\sup E$ is lower bound of F

(ii) If l is a lower bound of F, then $l \leqslant -\sup E$.

proofs

(i) $x \in F \Rightarrow -x \in E \Rightarrow -x \leqslant \sup E \Rightarrow x \geqslant -\sup E$.

(ii) $l \leqslant x \; \forall x \in F \Rightarrow -l \geqslant t \; \forall t \in E \Rightarrow -l \geqslant \sup E \Rightarrow l \leqslant -\sup E$.

□

Example. $\sup(-1, 2) = 2$, $\inf(-1, 2) = -1$.

3.8 $\sqrt{2}$ exists

Theorem. *There exists a real number α whose square is 2.*

Proof. Let $E = \{x \in \mathbb{R} \ : \ x^2 < 2\}$.
We split the proof into the following steps:

(a) *(i)* $1 \in E$;
(ii) $E \neq \emptyset$;
(iii) E *is bounded above by* 2.

(b) $\alpha := \sup(E)$ *exists.*

(c) $\alpha > 0$.

(d) *(i)* $\alpha^2 > 2$ *is impossible;*
(ii) $\alpha^2 < 2$ *is impossible.*

(e) $\alpha^2 = 2$.

Here are the proofs of these statements:

(a) (i) $1^2 = 1 < 1 + 1$ (follows from $0 < 1$).
(ii) Follows from above.
(iii) Take $x \in E$. If $x \leqslant 1$ then $x < 2$. If $x > 1$ then $x \leqslant x^2 < 2$.

(b) $\sup(E)$ exists by Completeness Axiom.

(c) $\alpha > 0$ since it exceeds $1 \in E$.

(d) (i) $\alpha^2 > 2$ *is impossible.*
Suppose that $\alpha^2 > 2$. Let $h > 0$, in fact let $h =$.
[*We will come back later and write in the value here, once we know what we need! This is a typical way to argue ...*]
As $\alpha - h < \alpha$ it is not the supremum, and so for some $e \in E$ we have $\alpha - h > e$; then $(\alpha - h)^2 < e^2 < 2$ so that $\alpha^2 - 2h\alpha + h^2 < 2$.
As $h^2 > 0$ this gives $\alpha^2 - 2h\alpha < 2$, and so—since $\alpha > 0$—$h > \frac{\alpha^2-2}{2\alpha}$.
[*We now see that if we had chosen h small enough we'd get a contradiction. So we go back to where we left a gap, and fill in the value $\frac{1}{2}\frac{\alpha^2-2}{2\alpha}$ before continuing our proof.*]
But as $h = \frac{1}{2}\frac{\alpha^2-2}{2\alpha}$ this is impossible.

(ii) $\alpha^2 < 2$ *is impossible.*
Suppose on the contrary that $\alpha^2 < 2$. Let $h > 0$; in fact let $h =$.
[*Same idea.*
Then $(\alpha + h)^2 = \alpha^2 + 2h\alpha + h^2 < \alpha^2 + 3h\alpha$, since $h < \alpha$.
[*We'll arrange for that to be true in a moment.*]
Hence we even get $(\alpha + h)^2 < \alpha^2 + 3h\alpha < 2$, provided $h < \frac{2-\alpha^2}{3\alpha}$.
[*So we go back and fill in the value $\frac{1}{2}\min(\alpha, \frac{2-\alpha^2}{3\alpha})$ for h. Then all we've said is OK and we continue.*]
Hence $\alpha + h \in E$ and since $\alpha = \sup(E)$ we get $\alpha + h \leqslant \alpha$, a contradiction.

(e) Finally, $\alpha^2 = 2$ follows from the Trichotomy Axiom P3.

□

Note. *We don't usually show all the 'scaffolding' in this way, we just show the finished proof. But if you're building a proof yourself you need to argue in this sort of way initially.*

Note. *We write* $\alpha = \sqrt{2}$.

3.9 A generalisation

If we have any $a \in \mathbb{R}$ with $a \geqslant 0$ then a similar argument establishes the existence of $\sqrt{a}$. We will use this fact from now on. An alternative proof will appear on the Exercise sheets.

3.10 A useful irrational

Corollary. *There is an irrationalin* $(0, 1)$.

Proof. If $\alpha \leqslant 2$ then $\alpha^2 \geqslant 4 > 2$; so $1 \leqslant \alpha \leqslant 2$. Therefore

$$\frac{1}{2} \leqslant \frac{1}{\alpha} \leqslant 1$$

so $\frac{1}{\alpha} \in (0, 1]$ and as $\alpha \neq 1$, $0 < \frac{1}{\alpha} < 1$. □

3.11 Approximation Property

Let E be bounded, $E \neq \emptyset$. Suppose $\varepsilon > 0$. Then $\sup E - \varepsilon < x$ for some $x \in E$.

Proof. If not, $\sup E - \varepsilon$ is an upper bound and $\sup E - \varepsilon > \sup E$ which is a contradiction. □

3.12 Archimedean Property

Let $x \in \mathbb{R}$. Then for some $n \in \mathbb{N}$, $x < n$.

Proof. If not, $\mathbb{N}$ is bounded and not empty. So let $\xi = \sup \mathbb{N}$. Then $\xi - 1 < \xi$, so $\xi - 1 < k$ for some $k \in \mathbb{N}$. Then $\xi < k + 1$ and as $k + 1 \in \mathbb{N}$, $k + 1 < \xi$. So $\xi < \xi$, which is contradiction to trichotomy. □

3.13 Small Rationals

Corollary. *Let* $x \in \mathbb{R}$, $x > 0$. *Then for some* $n \in \mathbb{N}$, $\frac{1}{n} < x$.

Proof. Apply Archimedean Property to $\frac{1}{x}$. □

In fully: there are big integers and very small rationals.

3.14 The rationals are everywhere

Corollary. *Let (a, b) be an interval. Then there exists a rational number in (a, b).*

Proof. (i)
Apply (3.13) to $b - a$ to find n such that $\frac{1}{n} < b - a$.

2. Apply (3.12) to b to get $N > b$.
3. Let $S := \{k \in \mathbb{N} : N - \frac{k}{n} > b\}$
4. $N - \frac{k+1}{n} < N - \frac{k}{n}$.
5. Hence either S has a maximum k_1, or $S = \mathbb{N}$.
6. In first case $N - \frac{k_1+1}{n}$ must be less than b and cannot (without contradicting choice of n be less than a. So we are done.
7. If for all k we get $N - \frac{k}{n} > b$ then $\mathbb{N}$ is bounded above by $(N - b)n$ contrary to the Archimedean Property.

□

3.15 $\mathbb{R}$ is not countable

If it was, so would $[0, 1]$ be countable. Clearly $[0, 1]$ not finite as it contains all $\frac{1}{k}$. So let $\theta : \mathbb{N} \to [0, 1]$ be a bijection and write $x_k := \theta(k)$.

- Define a_1, b_1 so that $x_1 \notin [a_1, b_1]$. (Question: How to do it?)
- Define a_2, b_2 so that $a_1 < a_2 < b2 < b_1$ and $x_2 \notin [a_2, b_2]$. (Question: How to do it?)
- . . . (Repeat this process.)

$E = \{a_j : j \in \mathbb{N}\}$ is bounded above by 1 and so let $\lambda = \sup E$.
$F = \{b_j : j \in \mathbb{N}\}$ is bounded below by 0 and so let $\mu = \inf F$.

Note. $a_j \leqslant a_{j+s} \leqslant b_{j+s} \leqslant b_j$

Hence for all m, n [look at $m < n$ and $n < m$] $a_m \leqslant b_n$. So b_n is upper bound of E, so $\lambda \leqslant b_n$ for all n. Similarly $\mu \geqslant a_m$ for all m. So λ is lower bound of F. So $\lambda \leqslant \mu$

$$a_n \leqslant \lambda \leqslant \mu \leqslant b_n \ \forall n$$

As $\lambda \geqslant a_{n+1} > a_n$, $a_n < \lambda \leqslant \mu < b_n \ \forall n$. So $\frac{\lambda+\mu}{2} \in [0, 1]$ and $\neq x_n$ for every n.

4 Complex numbers

4.1 Algebraic Properties

We can define $\mathbb{C}$ from $\mathbb{R}$: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.
We define $+$, $-$, 0, $\times$, $1/\cdot$, 1 by the usual rules. A1—A4, M1—M4, D, and Z are all true.

4.2 Order?

What about P1—P3? Impossible, see exercise sheet.
So what can we salvage? The answer is, the modulus function—so at least we can estimate the sizes of things ...

4.3 Definition and easy properties of $|\ |$

(i) $|z| := \sqrt{(x^2 + y^2)}$: makes sense as $x^2 + y^2 \geqslant 0$.

(ii) $\overline{z} := x - iy$

(iii) $|z| = |\overline{z}|$

(iv) $|z|^2 = z\overline{z}$

(v) $\Re z := x$, $\Im z := y$

(vi) $|\Re z| \leqslant |z|$, $|\Im z| \leqslant |z|$

(vii) $z + \overline{z} = 2\Re z$

(viii) $|zw| = |z||w|$

4.4 Δ Law

$|z + w| \leqslant |z| + |w|$

Proof. We use the above properties.

$$
\begin{aligned}
|z + w|^2 &= z\overline{z} + (z\overline{w} + \overline{z}w) + w\overline{w} \\
&= z\overline{z} + 2\Re z\overline{w} + w\overline{w} \\
&\leqslant z\overline{z} + 2|z\overline{w}| + w\overline{w} \\
&= z\overline{z} + |z\overline{w}| + |w\overline{z}| + w\overline{w} \\
&= (|z| + |w|)^2
\end{aligned}
$$

□

Now see exercise sheet 2.