ANALYSIS I

5 Real and complex sequences

5.1 Real numbers in practice

How do we get hold of a real number? Answer, we look at successive approximations. E.g.:

$$1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \ldots \longleftarrow \sqrt{2}$$

Our task is to make all this precise ...

5.2 Sequences

A **sequence** of real numbers is a function $\alpha : \mathbb{N} \longrightarrow \mathbb{R}$.
A **sequence** of complex numbers is a function $\alpha : \mathbb{N} \longrightarrow \mathbb{C}$.
E.g. $\sigma(n) = (-1)^n$, e.g. $\zeta(n) = 0$; e.g. $\iota(n) = n$ etc. etc.
Note, we usually just give the values, and say "the sequence $1, \frac{1}{2}, \frac{1}{4}, \ldots$" if it is clear what the function "must be". Or better we write "the sequence $(a_n)_{n=1}^{\infty}$" or "the sequence (a_n)".
Take care!

5.3 New sequences from old

Suppose (a_n) and (b_n) are sequences of real numbers and $c \in \mathbb{R}$. We define the sequences $(a_n + b_n)$, (ca_n), $(a_n b_n)$, (a_n/b_n) in the obvious way. All are well defined except possibly the quotient.

Example. *$a_n = (-1)^n$, $b_n = 1$ $\forall n$:*

$$\begin{aligned} (a_n + b_n) &= (0, 2, 0, 2, \ldots) \\ (-a_n) &= ((-1)^{n+1}) \\ (a_n b_n) &= (a_n) \end{aligned}$$

5.4 Tails

Let (a_n) be a sequence of real numbers and let $k > 0$. Define $b_n = a_{n+k}$ then (b_n) is a sequence. Usually we write $(a_{n+k})_{n=1}^{\infty}$. This we call the **tail** of the given sequence.

5.5 Definition of convergence

Let (a_n) be a sequence of real numbers, and let $l \in \mathbb{R}$. We say "(a_n) **converges** to l", and write $a_n \to l$ as $n \to \infty$, if for every positive real number $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that

$$n \geqslant N \implies |a_n - l| < \varepsilon$$

If this happens, we say l is **the limit** of (a_n) and we say that (a_n) **is convergent** if for some $l \in \mathbb{R}$, (a_n) converges to l.

Note. *Some people say "tends to". We also write* $\lim a_n = l$, $\lim_{n\to\infty} a_n = l$.

5.6 Examples

(i) $a_n = \frac{2^n-1}{2^n}$. Then $a_n \to 1$

Proof. Look at
$$|a_n - 1| = \left|\frac{2^n-1}{2^n} - 1\right| = \frac{1}{2^n}.$$
Given $\varepsilon > 0$, how do we find N? Well notice that $\frac{1}{2^n} < \frac{1}{n}$ [*Prove $n < 2^n$ by induction!*]. Use our Archimedean Property to find N such that $\frac{1}{N} < \varepsilon$. Then for $n > N$, $\frac{1}{n} < \frac{1}{N}$. So
$$|a_n - 1| \leqslant \frac{1}{2^n} \leqslant \frac{1}{n} < \frac{1}{N} < \varepsilon$$
$\square$

(ii) The sequence
$$a_n = \frac{n^2+n+1}{3n^2+4}$$
is convergent.

Proof. (a) How do we guess l? Well
$$\frac{1+\frac{1}{n}+\frac{1}{n^2}}{3+\frac{4}{n^2}} \approx \frac{1}{3}$$
NOT PART OF OUR PROOF

(b) Now suppose $\varepsilon > 0$

(c) Look at
$$\left|a_n - \frac{1}{3}\right| = \left|\frac{1+\frac{1}{n}+\frac{1}{n^2}}{3+\frac{4}{n^2}} - \frac{1}{3}\right| = \frac{\frac{3}{n}-\frac{1}{n^2}}{3\left(3+\frac{4}{n^2}\right)} = \frac{3n-1}{3(3n^2+4)}$$
$$\leqslant \frac{3n}{3(3n^2+4)} \leqslant \frac{3n}{3\cdot 3n^2} \leqslant \frac{1}{n}$$

(d) By Archimedean Property there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$

(e) $n \geqslant N \implies \frac{1}{n} \leqslant \frac{1}{N} \leqslant \varepsilon$ so done.

$\square$

(iii) Let
$$a_n = \frac{n^2+(-1)^n}{n^2+1}.$$

Proof. (a) Guess $a = 1$

(b) Let $\varepsilon > 0$

(c)
$$|a_n - 1| = \left|\frac{n^2+(-1)^n-n^2-1}{n^2+1}\right| \leqslant \frac{2}{n^2+1} \leqslant \frac{1}{n}$$

(d) By Archimedean Property there exists N such that $N > \frac{1}{\varepsilon}$

(e) Done

$\square$

5.7 Complex Sequences

Let (z_n) be a sequence of complex numbers and let $w \in \mathbb{C}$. We say that (z_n) converges to w and write $z_n \to w$ (or $\lim z_n = w$ etc.) if for every positive real number $\varepsilon > 0$, there exists a natural number N such that

$$n \geqslant N \implies |z_n - w| < \varepsilon$$

Theorem. *Let $z_n = x_n + iy_n$.*

(i) $z_n \to z \implies x_n \to \Re z,\ y_n \to \Im y$

(ii) $x_n \to x,\ y_n \to y \implies z_n \to x + iy$

Proof. (i) Put $x = \Re z$. $|x_n - X| = \Re(z_n - z) \leqslant |z_n - z|$. So given $\varepsilon > 0$ use the same N.

(ii)

$$|z_n - z| \leqslant |x_n - x| + |y_n - y| \text{ by } \Delta \text{ law}$$

Find N_1 to ensure first term is less than $\varepsilon/2$, and N_2 to ensure second is less than $\varepsilon/2$—then use $N := \min(N_1, N_2)$.

□

Note. *A 2ε argument.*

5.8 Example

Let $z_n = \left(\frac{1}{1+i}\right)^n$. Then $z_n \to 0$.

Proof.

$$|z_n - 0| = \left|\left(\frac{1}{1+i}\right)^n\right| = \left|\frac{1}{1+i}\right|^n = \frac{1}{|1+i|^n} = \frac{1}{|\sqrt{2}|^n} = \frac{1}{2^{n/2}} \leqslant \underbrace{\frac{1}{n} < \varepsilon}_{\text{Arch.}}$$

□

5.9 Uniqueness of Limits

Theorem. *suppose that $a_n \to e_1$ and $a_n \to e_2$. Then $e_1 = e_2$.*

Proof. Suppose $\varepsilon > 0$. Then $\varepsilon/2 > 0$ so there exists N_1 such that

$$n \geqslant N_1 \implies |a_n - e_1| < \varepsilon/2 \tag{†}$$

Again $\varepsilon/2 > 0$ so there exists N_2 such that

$$n \geqslant N_2 \implies |a_n - e_2| < \varepsilon/2. \tag{‡}$$

Suppose now $n \geqslant \max(N_1, N_2)$. Then

$$\begin{array}{rcccccl} |e_1 - e_2| & \leqslant & |e_1 - a_n| & + & |e_2 - a_n| & \text{by the } \Delta \text{ law} \\ & < & \varepsilon/2 & + & \varepsilon/2 & = \varepsilon \end{array}$$

If $e_1 \neq e_2$ then $|e_1 - e_2| > 0$. Chose $\varepsilon = |e_1 - e_2|$ which contradicts to $|e_1 - e_2| < \varepsilon$ by (†) and (‡). □

5.10 The secret of success

Compare the sequence you're looking at with ones you already know about.

5.11 Inequalities preserved

Theorem. *Let (a_n) and (b_n) be real sequences, which $a_n \to a$ and $b_n \to b$. Suppose that $a_n \leqslant b_n$ for all n. Then $a \leqslant b$.*

Proof. Suppose, as the contrary, that $b < a$. Then $a - b > 0$. Chose $\varepsilon = \frac{a-b}{2} > 0$. Then there exists

$$\begin{aligned} N_1 \text{ such that } n \geqslant N_1 &\implies |a_n - a| < \varepsilon \\ N_2 \text{ such that } n \geqslant N_2 &\implies |b_n - b| < \varepsilon \end{aligned}$$

So

$$\begin{aligned} n \geqslant N_1 &\implies a - \varepsilon < a_n < a + \varepsilon \\ n \geqslant N_2 &\implies b - \varepsilon < b_n < b + \varepsilon \end{aligned}$$

So for $n \geqslant \max(N_1, N_2)$

$$\left.\begin{aligned} a_n > a - \varepsilon > a - \frac{a-b}{2} = \frac{a+b}{2} \\ b_n < b + \varepsilon < b + \frac{b-a}{2} = \frac{3b-a}{2} \end{aligned}\right. \qquad (\text{as } a > b)$$

Therefore $a_n > b_n$ which is contradiction. □

5.12 A corollary

Corollary. *Suppose $a_n \to a$, $b_n \to b$ and $a_n \leqslant b_n$ for all $n \geqslant K$. Then $a \leqslant b$.*

Proof. For you. □

5.13 Sandwich rule

Suppose $x_n \to a$ and $y_n \to a$. If $x_n \leqslant a_n \leqslant y_n$ then $a_n \to a$.

5.14 The tail wags the dog

Theorem.

Let (a_n) be a sequence, and let $k \in \mathbb{N}$. Put $b_n = a_{n+k}$. Then the followings are equivalent:

(i) $a_n \to l$

(ii) $b_n \to l$

Proof.

(i)⇒(ii) Let $\varepsilon > 0$. Then there exists N_1 such that $n \geqslant N_1 \Longrightarrow |a_n - l| < \varepsilon$. With $N_2 = N_1$

$$n \geqslant N_2 \Longrightarrow n + k \geqslant N_1 \Longrightarrow |a_{n+k} - l| < \varepsilon \Longrightarrow |b_n - l| < \varepsilon$$

So $b_n \to l$.

(ii)⇒(i) Let $\varepsilon > 0$. Then there exists N_2 such that $n \geqslant N_2 \Longrightarrow |b_n - l| < \varepsilon$. With $N_1 = N_2 + k$

$$n \geqslant N_1 \Longrightarrow (n - k) \geqslant N_2 \Longrightarrow |b_{n-k} - l| < \varepsilon \Longrightarrow |a_n - l| < \varepsilon.$$

□

5.15 Some notation

Let a_n, b_n be sequences. We write $a_n = O(b_n)$ if there exist c such that for some N

$$n \geqslant N \Longrightarrow |a_n| < cb_n$$

We write $a_n = o(b_n)$ if $\frac{a_n}{b_n}$ is defined and

$$\frac{a_n}{b_n} \to 0$$

Example. *(i)* $n = O(n^2)$

(ii) $n = o(n^2)$

(iii) $\sin n\theta = O(1)$

(iv) $sinn\theta = o(n)$

6 What it's about

6.1 The "Algebra of limits"

Most sequences can be built up from simpler ones using the addition, multiplication, and so on. The algebra of limits tells us how the limits behave.

6.2 AOL: Constants

If $a_n = a$ for all n then $a_n \to a$.

Proof. Take $N = 1$; $n \geqslant N \implies |a_n - a| = 0 < \varepsilon$ $\square$

6.3 AOL: Sums

If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n + b_n \to a + b$.

Proof. Let $\varepsilon > 0$ Then $\varepsilon/2 > 0$. So

$$\begin{array}{rcl} \exists N_1 \; : \; n \geqslant N_1 & \implies & |a_n - a| < \varepsilon/2 \\ \exists N_2 \; : \; n \geqslant N_2 & \implies & |b_n - b| < \varepsilon/2 \end{array}$$

Put $N_3 = \max(N_1, N_2)$. Then

$$\begin{array}{rcccl} n \geqslant N_3 \implies & & |(a_n + b_n) & - & (a + b)| \\ & \leqslant & |a_n - a| & + & |b_n - n| \quad \text{by the } \Delta \text{ law} \\ & < & \varepsilon/2 & + & \varepsilon/2 \\ & = & \varepsilon & & \end{array}$$

[A 2ε-proof] $\square$

6.4 AOL: Differences

If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n - b_n \to a_b$.

Proof. You do it! $\square$

6.5 AOL: Translation

If $_n \to a$ as $n \to \infty$ then $a_n - c \to a - c$.

Proof. $|(a_n - c) - (a - c)| = |a_n - a|$ so proof is done. $\square$

6.6 AOL: Scalar product

If $a_n \to a$ as $n \to \infty$ and $\lambda \in \mathbb{R}$ then $\lambda a_n \to \lambda a$.

Proof. Let $\varepsilon > 0$. Then there exists $N : |a_n - a| < \varepsilon$ for all $n \geqslant N$ and

$$|\lambda a_n - \lambda a| = |\lambda||a_n - a| \leqslant |\lambda|\varepsilon$$

for all $n \geqslant N$. So case

(i) if $\lambda = 0$ then $\lambda a_n = 0$, $\lambda a = 0$.

(ii) if $\lambda \neq 0$ then $\varepsilon/|\lambda| > 0$ so there exists N such that $n \geqslant N \Longrightarrow |a_n - a| < \varepsilon/|\lambda|$. Then $|\lambda a_n - \lambda a| \leqslant |\lambda|\varepsilon/|\lambda| = \varepsilon$.

$\square$

6.7 AOL: Products

If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n b_n \to ab$.

Proof.

$$a_n b_n - ab = (a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b)$$

Lemma. *If $x_n \to 0$ and $y_n \to 0$ then $x_n y_n \to 0$.*

Proof of Lemma. Given $\varepsilon > 0$, let $\varepsilon_1 = \min(1, \varepsilon)$ then

$$\begin{aligned} \exists N_1,\ n \geqslant N_1 &\implies |x_n| < \varepsilon_1 \\ \exists N_2,\ n \geqslant N_2 &\implies |y_n| < \varepsilon_1 \end{aligned}$$

Then

$$n \geqslant \max(N_1, N_2) \implies |x_n y_n| \leqslant |x_n||y_n| \leqslant \varepsilon_1^2 < \varepsilon_1 \leqslant \varepsilon$$

$\square$

Now

$$\left.\begin{array}{ll} a_n - a \to 0 & \text{by (6.5)} \\ b_n - b \to 0 & \text{by (6.5)} \end{array}\right\} \Longrightarrow (a_n - a)(b_n - b) \to 0 \text{ by Lemma}$$

and

$$\begin{array}{ll} b(a_n - a) \to 0 & \text{by (6.6)} \\ a(b_n - b) \to 0 & \text{by (6.6)} \end{array}$$

Then the proof is completed by (6.3). $\square$

6.8 AOL: Reciprocals

If $a_n \to a_n 0$ and $a_n \neq 0$ for all n. Then $\frac{1}{a_n} \to \frac{1}{a}$.

Needs care. We prove it first for the case $a > 0$.

(i) As $a/2 > 0$ there exists N_1 such that

$$\begin{aligned} n \geqslant N_1 \Longrightarrow |a_n - a| < a/2 &\implies a_n > a/2 \\ &\implies \frac{1}{a_n} < \frac{2}{a} \end{aligned}$$

(ii) Now let $\varepsilon > 0$; so $a^2 \frac{\varepsilon}{2} > 0$. So there exist N_2 such that

$$n \geqslant N_2 \Longrightarrow |a_n - a| < a^2 \frac{\varepsilon}{2}$$

(iii) Put $N_3 = \max(N_1, N_2)$. Then

$$n \geqslant N_3 \Longrightarrow \left| \frac{1}{a_n} - \frac{1}{a} \right| \leqslant \frac{|a_n - a|}{|a_n||a|} \leqslant \left(a^2 \frac{\varepsilon}{2} \right) \frac{2}{a} \frac{1}{a} = \varepsilon$$

Now if $a < 0$ we can deduce the result from the above and scalar multiplication by $\lambda = -1$. $\square$

6.9 AOL: Quotients

If $a_n \to a$, $b_n \to b \neq 0$ and $b_n \neq 0$ for all n. Then $a_n/b_n \to a/b$

Proof. (6.8) and (6.7). ☐

6.10 AOL: Modulus

If $a_n \to a$ then $|a_n| \to |a|$.

Proof.

$$\big||a_n| - |a|\big| \leqslant |a_n - a|$$

by Exercise sheet 1. ☐

6.11 Examples

(i)

$$\frac{n^2+n+1}{3n^2+4} \to \frac{1}{3}$$

Proof.
- $\frac{1}{n} \to 0$ [Proof: By Arch.]; $1 \to 1$, $3 \to 3$
- $\frac{1}{n^2} \to 0$ by (6.7)
- $1 + \frac{1}{n} + \frac{1}{n^2} \to 1$ by $(6.3)^2$
- $3 + \frac{4}{n^2} \to 3$ by (6.3)
- $\frac{1}{3+\frac{4}{n^2}}$ by (6.8)

☐

(ii) Suppose $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = a_{n+1} + a_n$, $n \geqslant 1$ [*Fibonacci numbers.*] Then (a_n) is convergent.

Proof. By induction, $a_n \geqslant 1$ for all n.So for $n \geqslant 1$

$$\left(\frac{a_{n+2}}{a_{n+1}}\right) = 1 + \left(\frac{a_n+1}{a_n}\right)^{-1}$$

Write $x_n = \frac{a_{n+1}}{a_n}$ for $n \geqslant 1$. $x_n >$ for all n. Then $x_1 = 2$ and $x_{n+1} = 1 + \frac{1}{x_n}$.
Suppose $x_n \to x$. By Tails: $x_{n+1} \to x$ and $1 + \frac{1}{x_n} \to 1 + \frac{1}{x}$ by AOL. So $x = 1 + \frac{1}{x}$ by Uniqueness of Limits. So $x^2 - x - 1 = 0$. So $x = \frac{1 \pm \sqrt{5}}{2}$. But $x_n > 0$ for all n, and so $x > 0$ by 5.11. So $x = \frac{1+\sqrt{5}}{2} > 1$.
Now

$$x_{n+1} - \underbrace{\frac{1+\sqrt{5}}{2}}_{=:\tau} = 1 + \frac{1}{x_n} - \tau = 1 + \frac{1}{x_n} - 1 - \frac{1}{\tau} = \frac{1}{x_n} - \frac{1}{\tau}$$

as $\tau^2 = \tau + 1$. So

$$\left|\frac{x_{n+1} - \tau}{x_n - \tau}\right| = \frac{1}{|x_n||\tau|} = \frac{1}{\tau x_n} \leqslant \frac{1}{\tau}$$

Then

$$\underbrace{-\frac{1}{\tau^2}}_{\to 0} \leqslant (x_{n+1} - \tau) \leqslant \underbrace{\frac{1}{\tau^{n-1}}}_{\to 0}$$

so we get $x_{n+1} \to \tau$ by Sandwich rule and $x_n \to \tau$ by Tail. $\square$