ANALYSIS I

7 Monotone Sequences

7.1 Definitions

We begin by a definition. We say that a real sequence (a_n) is

monotone increasing if	$n_1 < n_2 \Longrightarrow a_{n_1} < a_{n_2}$
monotone decreasing if	$n_1 < n_2 \Longrightarrow a_{n_1} > a_{n_2}$
monotone non-decreasing if	$n_1 < n_2 \Longrightarrow a_{n_1} \leqslant a_{n_2}$
monotone non-increasing if	$n_1 < n_2 \Longrightarrow a_{n_1} \geqslant a_{n_2}$

Example. *Let $a_n = n$. Then (a_n) is monotone increasing. So is $a_n = (2n+1)^2$.*

7.2 Decimals

Let $\alpha_n =$ largest integer less then $10^n(1/\sqrt{2})$. [Arch.]
Let $a_n = \frac{\alpha_n}{10^n}$. Then (a_n) is monotone non-decreasing since $10\alpha_n \leqslant \alpha_{n+1}$.
Question: Does $\lim a_n$ exist? Well,

$$0 \leqslant \frac{1}{\sqrt{2}} - a_n = \frac{1}{\sqrt{2}} - \frac{\alpha_n}{10^n} \leqslant \frac{1}{10^n}$$

so by Sandwich Rule $a_n \to \frac{1}{\sqrt{2}}$.
Decimal expansions exist! What we now want to do is to show that all 'bounded' monotone increasing sequences are convergent.

7.3 Bounded sequences

We say that a real sequence (a_n) is bounded above if the set $S := \{a_n : n \in \mathbb{N}\}$ is bounded above. Similarly (a_n) is bounded below if the set S is bounded below and (a_n) is bounded if S is bounded.

7.4 Bounded Sequences converge

Theorem. *Let (a_n) be a bounded above monotone non-decreasing sequence. Then (a_n) is convergent.*

Proof. Let $l = \sup\{a_n : n \in \mathbb{N}\}$; exists by C. Let $\varepsilon > 0$. Then $l - \varepsilon < l$, so is not sup. Hence for some $N \in \mathbb{N}$, $l - \varepsilon < a_N$. Now for $n \geqslant N$,

$$\begin{aligned} l - \varepsilon < a_N \leqslant \ & a_n \\ & a_n \ \leqslant l \end{aligned}$$

So

$$n \geqslant N \Longrightarrow |a_n - l| < \varepsilon$$

i.e. $a_n \to l$. □

7.5 Bounded sequences converge (the other case)

Corollary. *If (a_n) is bounded below and monotone non-increasing, then a_n tends to the infimum of $\{a_n \ : \ n \in \mathbb{N}\}$.*

7.6 An example: $2\cos\frac{\pi}{2^{n+1}}$

Let $a_1 = \sqrt{2}$ and for all $n \geqslant 0$ let $a_{n+1} = \sqrt{2+a_n}$

(0) By induction: $a_n \geqslant 0$ for all n.

(i) (a_n) is monotone:

Note that $a_{n+2}^2 - a_{n+1}^2 = 2 + a_{n+1} - 2 - a_n = a_{n+1} - a_n$.

So prove by induction: $a_{n+1} \geqslant a_n$.

The root is $\sqrt{2+\sqrt{2}} \geqslant 2$; the inductive step is what we noted above.

(ii) (a_n) is bounded above:

Well $a_1 < 2$, so $a_2 = \sqrt{2+a_1} \leqslant \sqrt{2+2} = 4$. Then by induction: for all n, $a_n \leqslant 2$.

(iii) So $a_n \to l$.

By	Tails	$a_{n+1} \to l$
	AOL	$a_{n+1}^2 \to l^2$
	AOL	$a_n + 2 \to l + 2$
	Uniqueness of limits	$l^2 = l + 2$

So $l = -1$ or $l = 2$. But $a_n \geqslant 0$ therefore $l \geqslant 0$ i.e. $l = 2$.

8 Subsequences

8.1 Informal Examples

Informal example: let $a_n = \frac{1}{n^2}$ i.e.

$$(a_n) = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots\right)$$

We can get new sequences by looking at

everything after second place	$\left(\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots\right)$	Tails
all odd terms	$\left(1, \frac{1}{9}, \frac{1}{25}, \dots\right)$	
all prime terms	$\left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{25}, \frac{1}{49}, \dots\right)$	
etc.		

8.2 Definition

Let (a_n) be a sequence. We call (b_n) a subsequence of (a_n) if $b_n = a_{f(n)}$ for some (strictly) monotone increasing function f. Usually we just say $(a_{n_r})_{r=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$ using the sequence notation $r \mapsto n_r$ for our increasing function $\mathbb{N} \longrightarrow \mathbb{N}$.

Note. *We don't pick **values** we pick **places** in sequence.*

8.3 Convergence of Subsequences of a Convergent Sequence

Theorem. *Suppose that the sequence $(a_n) \to l$. Then every subsequent $(a_{n_r}) \to l$.*

Proof. Let $\varepsilon > 0$. Then there exist N such that

$$n \geqslant N \Longrightarrow |a_n - l| < \varepsilon$$

As $r \longrightarrow n_r$ is increasing, $n_r \geqslant r$ (*Proof:* by induction on r. $n_{r+1} > n_r \geqslant r$, and $n_{r+1} \in \mathbb{N}$.) So

$$r \geqslant N \Longrightarrow n_r \geqslant N \Longrightarrow |a_{n_r} - l| < \varepsilon.$$

□

8.4 Converse?

The falsity of most forms of the converse is left for the exercise sheets.

8.5 Bolzano-Weierstrass Theorem

Theorem. *Let (a_n) be a bounded sequence of real numbers. Then (a_n) has a convergent subsequence.*

Proof.

(A) We show that (a_n) has a monotone subsequence.

We look at $V := \{k \in \mathbb{N} \ : \ m > k \Longrightarrow a_m < a_k\}$ the set of "viewpoints" . So at (k, a_k) we get a view all the way to ∞— "from here to infinity".

Case (a) $|V|$ is infinite.

Then we can list the elements of V in increasing order: $k_1 < k_2 < \dots$ and then (a_{k_r}) is subsequence and

$$r > s \Longrightarrow k_r > k_s \Longrightarrow a_{k_r} < a_{k_s}$$

i.e. (a_{k_r}) monotone decreasing.

Case (b) $|V|$ is finite.

Choose $m_1 >$ every element of V.

- As $m_1 \notin V$ there exists m_2 such that $a_{m_2} \not< a_{m_1}$ i.e. $a_{m_2} \geqslant a_{m_1}$
- As $m_2 \notin V$ there exists m_3 such that $a_{m_3} \geqslant a_{m_2}$
- ...

Continue in this way and define m_j for all j. Then $(a_{m_s})_{s=1}^{\infty}$ is monotone non-decreasing sequence.

(B) Now use the result on bounded monotone sequences to show our subsequence is convergent.

□

8.6 Many, many subsequences

Example. *Let $n = 2^{e_1}3^{e_2}\dots$ as prime factors. Let*

$$a_n = \frac{e_1}{1+e_1+e_2}.$$

Then

n	e_1	e_2	$1+e_1+e_2$	a_n
1	0	0	1	0
2	1	0	2	$\frac{1}{2}$
3	0	1	2	0
4	2	0	3	$\frac{2}{3}$
5	0	0	1	0
6	1	1	3	$\frac{1}{3}$
7	0	0	1	0
8	3	0	4	$\frac{3}{4}$
9	0	2	3	0
10	1	0	2	$\frac{1}{2}$
11	0	0	1	0
12	2	1	4	$\frac{1}{2}$
$\vdots$				

Actually $V = \emptyset$, and

$$n = 2^{e_1}3^{e_2}m \Longrightarrow a_n = \frac{e_1}{1+e_1+e_2} = \frac{2e_1}{2+2e_1+2e_2} < \frac{2e_1+1}{1+(2e_1+1)+(2e_2)} = a_N$$

for $N := 2^{2e_1+1}3^{2e_2}m > n$. If we take subsequence $n_k = 2^{\alpha 2^k + (2^k-1)}3^{\beta 2^k}$. Then

$$a_{n_k} \to \frac{\alpha+1}{\alpha+\beta+1}$$

8.7 Limit points

Let S be a set. We say that x is a **limit point** of S if for all $\varepsilon > 0$,

$$(x - \varepsilon, x + \varepsilon) \cap S \setminus \{x\} \neq \emptyset.$$

Example.

(i) 1 *is a limit point of* $(0, 1)$.

(ii) 1 *is not a limit point of* $\mathbb{N}$