ANALYSIS I

11 Some Tests for Convergence

11.1 Easy Observation

Theorem. *If* $\sum a_n$ *is convergent then* $a_n \to 0$.

Proof. Let $\varepsilon > 0$. By Cauchy Criterion there exists N such that

$$l \geqslant k \geqslant N \Longrightarrow \left| \sum_k^l a_n \right| < \varepsilon$$

In particular $\left| \sum_l^l a_n \right| < \varepsilon$, i.e. $|a_l| < \varepsilon$. □

What about the converse? Oh, very false: look at Harmonic Series.

11.2 The Comparison Test

Theorem. *Suppose* a_n*,* b_n *sequences such that* $0 \leqslant a_n \leqslant b_n$*. Then*

(i) $\sum b_n$ is convergent $\Longrightarrow \sum a_n$ is convergent

(ii) $\sum a_n$ is divergent $\Longrightarrow \sum b_n$ is divergent

Proof. $(i) \Longrightarrow (ii)$ by Logic so it is enough to prove (i).
Let $\varepsilon > 0$. Then there exist N such that

$$l \geqslant k \geqslant N \Longrightarrow \sum_k^l b_n = \sum_k^l |b_n| < \varepsilon$$

Then

$$\sum_k^l |a_n| = \sum_k^l a_n \leqslant \sum_k^l b_n < \varepsilon$$

□

11.3 Examples

(i) $\sum_1^\infty \frac{1}{n^{\frac{5}{2}}}$ is convergent.

Proof. Compare with $\sum \frac{1}{n^2}$. □

(ii) $\sum \frac{1}{n(n+1)(2+\cos n)}$ is convergent.

Proof. Compare with $\sum \frac{1}{n^2}$. □

(iii) $\sum \frac{x^{n+1}}{n+1}$ is convergent for $-1 < x < 1$.

Proof. We use "Absolutely convergence $\Longrightarrow$ Convergence", and compare $\sum \frac{|x|^{n+1}}{n+1}$ with $\sum |x|^n$ □

11.4 A mild extension of Comparison Test

By Tails only needed $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant K$.

11.5 The Ratio Test

Let (a_n) be a real sequence, $a_n > 0$. Suppose that $\lim \frac{a_{n+1}}{a_n} = l$ exists. Then

(i) $l < 1 \implies \sum a_n$ is convergent

(ii) $l > 1 \implies \sum a_n$ is divergent

(iii) $l = 0$ tells us nothing

Proof.

(i) Chose x such that $l < x < 1$. With $\varepsilon = x - l > 0$ find N such that

$$n \geqslant N \implies \left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon$$

i.e.

$$\frac{a_{n+1}}{a_n} = \underbrace{\left| \frac{a_{n+1}}{a_n} \right| \leqslant \varepsilon + l}_{\text{by the } \Delta \text{ law}} = x$$

So by induction, $\frac{a_{N+k}}{a_N} \leqslant x^k$. Now $\sum x^k$ is convergent, so $a_N \sum x^k$ is convergent, so $\sum_{n=N}^{\infty} a_n$ is convergent by Comparison Test, so $\sum a_n$ is convergent by Tails.

(ii) Chose x such that $1 < x < l$ and find N such that

$$n \geqslant N \implies \frac{a_{n+1}}{a_n} > x$$

Then $a_{N+k} \geqslant x^k a_N$ and divergent by Comparison Test.

(iii) *Ex* (i) $a_n = \frac{1}{n+1}$, $\frac{a_{n+1}}{a_n} \to 1$ by AOL, and $\sum \frac{1}{n+1}$ is divergent.

Ex (ii) $a_n = \frac{1}{(n+1)^2}$, $\frac{a_{n+1}}{a_n} \to 1$ by AOL, and $\sum \frac{1}{(n+1)^2}$ is convergent.

□

11.6 Examples

(i) For all x, $\sum \frac{x^n}{n!}$ is convergent.

Proof. We will use "Absolute Convergence $\implies$ Convergence". So XXX, $x \geqslant 0$. [case $x = 0$ is trivial.]

$$\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0 \text{ as } n \to \infty \text{ and } 0 < 1$$

□

(ii) $\sum_{n=0}^{\infty} (n+1)x^n$ is convergent if $|x| < 1$.

Proof. The case $x = 0$ is trivial. For $x < 0$ use "Absolutely Convergence $\Longrightarrow$ Convergence".

If $0 < x < 1$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)x^{n+1}}{(n+1)x^n} \to x \text{ by AOL}$$

So convergent by Ratio Test. □

11.7 More notation

Let (a_n) be a real sequence, $a_n > 0$.
We write $a_n \to \infty$ as $n \to \infty$ to mean that given any real number B we can find a natural number N such that

$$n \geqslant N \Longrightarrow a_n > B.$$

11.8 Extension of Ratio Test

Theorem. *Let (a_n) be a real sequence, $a_n > 0$, and $\frac{a_{n+1}}{a_n} \to \infty$ as $n \to \infty$. Then $\sum a_n$ is divergent.*

Proof. The proof of the Ratio Test is easily adapted. □

11.9 Leibniz Alternating Series Test

Theorem. *Let (a_n) be a real series and suppose that (a_n) is monotone non-increasing with limit 0. Then $\sum(-1)^n a_n$ is convergent.*

Proof. Grouping the terms we can see that

$$\begin{aligned} 0 &\leqslant (a_k - a_{k+1}) + (a_{k+2} - a_{k+3}) + \dots + (a_{k+2d} - a_{k+2d+1}) \\ &= a_k - (a_{k+1} - a_{k+2}) - \dots - (\dots - a_{k+2d}) - a_{k+2d+1} \\ &\leqslant a_k \end{aligned}$$

and so

$$\begin{aligned} 0 &\leqslant a_k - a_{k+1} + \dots - a_{k+2d+1} + a_{k+2d+2} \\ &\leqslant a_k + a_{k+2d+1} \\ &\leqslant 2a_k \end{aligned}$$

so for all $l \geqslant k$,

$$\left| \sum_k^l (-1)^n a_n \right| \leqslant 2a_k.$$

Now we use $a_k \to 0$ and Cauchy for result. □

11.10 Estimating the Sum

Corollary. *Suppose that we have the same properties as above. Then*

$$\sum_{n=0}^{2N+1} (-1)^n a_n \leqslant \sum_{n=0}^{\infty} (-1)^n a_n \leqslant \sum_{n=0}^{2N} (-1)^n a_n.$$

Proof. Exercise. □

12 The Integral Test

12.1 Not a circular argument

Nothing we have done so far lets us tackle $\sum \frac{1}{n(\log n)^2}$ or evaluate $\sum \frac{(-1)^n}{n+1}$.
In this section we deal with these: but in order to do so we need to make use of the properties of "integral" and "logarithm" which are not established until the HT and TT courses.
At the end of the year you can persuade yourself that these properties which we are to now use do not depend on any of the results of this section.

12.2 The Integral Test

Theorem. *Let $f : [1, \infty) \longrightarrow [1, \infty)$ be continuous and decreasing. Let*

$$\delta_n := \sum_{k=1}^{n-1} f(k) - \int_1^n f(t)\mathrm{d}t$$

Then

(i) $\delta_n \geqslant 0$

(ii) $\delta_n \leqslant f(1)$

(iii) $\delta_{n+1} \geqslant \delta_n$

and so (δ_n) is convergent.

Corollary. *With f as above, $\sum_1^\infty f(k)$ is convergent if and only if $\lim_{n\to\infty} \int_1^n f(t)\mathrm{d}t$ exists.*

Proof of Corollary. AOL. □

12.3 Application of Integral Test

(i) $a_n := \frac{1}{n^\alpha}$ Hence $f(x) = \frac{1}{x^\alpha}$ and for $\alpha \geqslant 0$. $[f'(x) = -\alpha/x^{\alpha+1} < 0]$

$$\int_1^n f(t)\mathrm{d}t = \begin{cases} \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_1^n = \frac{1}{1-\alpha} - \frac{1}{(1-\alpha)n^{\alpha-1}} & \alpha \neq 1 \\ \log n & \alpha = 1 \end{cases}$$

If $\alpha > 1$, then convergent. If $0 \leqslant \alpha \leqslant 1$, then divergent.

Check also: $\alpha < 0$, then divergent $[a_n \nrightarrow 0]$.

(ii) $a_n := \frac{1}{n \log n}$ for $n \geqslant 2$. Hence $f(x) = \frac{1}{x \log x}$, and

$$f'(x) = \frac{-1}{x^2 \log x} - \frac{1}{x(\log x)^2}\frac{1}{x} = -\frac{1}{x^2(\log x)^2}[\log x - 1] < 0 \ x \geqslant 3$$

Then

$$\int_2^n \frac{1}{x \log x}\mathrm{d}x = \log\log n - \log\log 2 \to \infty$$

Therefore $\sum \frac{1}{n \log n}$ is divergent.

12.4 The Proof

As f decreasing,

$$f(k+1) \leqslant f(x) \leqslant f(k), \text{ if } \ k \leqslant x \leqslant k+1$$

As $\int$ XXX $\leqslant$, and $\int_n^{n+1} 1 = 1$

$$f(k+1) \leqslant \int_k^{k+1} f(x) \leqslant f(k)$$

Adding these

$$f(2) + f(3) + \cdots + f(n) \leqslant \int_1^n f(t)dt \leqslant f(1) + f(2) + \cdots + f(n-1)$$

So

$$0 \leqslant \sum_{r=1}^{n-1} f(r) - \int_1^n f(t)dt$$

(i) i.e. $1 \leqslant \delta_n$.

Also

$$\begin{aligned} \delta_n &= \textstyle\sum_{r=1}^{n-1} f(r) - \int_1^n f(t)dt \\ &\leqslant \textstyle\sum_{r=1}^{n-1} f(r) - \sum_{r=2}^{n} f(r) \\ &= f(1) - f(n) \qquad \leqslant f(1) \end{aligned}$$

(ii) Also

$$\delta_{n+1} - \delta_n = f(n) - \int_n^{n+1} f(t)dt \geqslant 0$$

Now (δ_n) bounded and monotone increasing and so convergent.

12.5 Euler's Number

Apply to $f(x) = \frac{1}{x}$ and get

$$\begin{aligned} \gamma_n &= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \textstyle\int_1^{n+1} \tfrac{dx}{x} && \text{is convergent} \\ &= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \log(n+1) && \text{is convergent} \end{aligned}$$

Call limit γ:

$$\gamma = \lim_{n\to\infty} \left(\sum \frac{1}{n} - \log(n+1) \right)$$

[Check $\lim(1 + \cdots + 1/n - \log n) \to \gamma$]

12.6 Application

$$\sum_{n=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2$$

Proof. We have already established convergence.

$$
\begin{array}{rcllllllll}
s_{2n} & = & 1-\frac{1}{2} & + & \frac{1}{3}-\frac{1}{4} & + & \dots & -\frac{1}{2n} & & \\
 & = & 1+\frac{1}{2} & + & \frac{1}{3}+\frac{1}{4} & + & \dots & +\frac{1}{2n} & - \log 2n & + \log 2n \\
 & & -1 & & -\frac{1}{2} & & \dots & -\frac{1}{n} & + \log n & - \log n \\
 & = & \multicolumn{8}{l}{\gamma_{2n}-\gamma_n+\log 2} \\
s_{2n+1} & = & \multicolumn{8}{l}{\gamma_{2n}-\gamma_n+\log 2+\frac{1}{2n+1}}
\end{array}
$$

As $n \to \infty$, both tend to $\log 2$. Easy to see $s_n \to \log 2$.

□