ANALYSIS I

13 Power Series

13.1 Definition

Let (a_n) be a real or complex series.
The series generated by the sequences $(a_n z^n)$ as z varies are called the power series generated by (a_n).
We usually just speak of 'the power series $(a_n z^n)$.

Note. *These are the most important series of all! (Taylor, Maclaurin, etc, etc.)*

Note. *We will prove our theorems for real (a_n) but almost all apply equally to real and complex sequences.*

13.2 Examples

(i) $a_n = 1 : \sum x^n$: Geometric Series : $\begin{array}{ll} \text{convergent} & \text{if } |x| < 1 \\ \text{divergent} & \text{if } |x| \geqslant 1 \end{array}$

(ii) $a_n = 1/n! : \sum x^n/n!$: Exponential Series : convergent for all x

(iii) $a_n = 1/n : \sum_{n/1}^{\infty} x^n/n$: Logarithmic Series : $\begin{array}{ll} \text{convergent} & \text{if } |x| < 1 \\ \text{divergent} & \text{if } |x| > 1 \end{array}$

(iv) $\left.\begin{array}{rcl} a_{2n} & = & (-1)^n/2n! \\ a_{2n+1} & = & 0 \end{array}\right\} : \sum \frac{x^{2n}(-1)^n}{2n!}$: Cosine Series : convergent for all x

(v) $\left.\begin{array}{rcl} a_{2n} & = & 0 \\ a_{2n+1} & = & (-1)^n/(2n+1)! \end{array}\right\} : \sum \frac{x^{2n+1}(-1)^n}{(2n+1)!}$: Sine Series : convergent for all x

13.3 The Circle of Convergence

Theorem. *If $\sum a_n x_0^n$ is convergent, and $|x| < |x_0|$ then $\sum a_n x^n$ is absolutely convergent.*

Proof. Chose p such that $\left|\frac{x}{x_0}\right| < p < 1$. As $\sum a_n x_0^n$ is convergent, $a_n x_0^n \to 0$, so there exist N such that

$$n \geqslant N \Longrightarrow |a_n x_0^n| < 1$$

Then

$$n \geqslant N \Longrightarrow |a_n x^n| \leqslant |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leqslant p^n$$

Then by Comparison Test with $(1, 1, 1, \ldots, 1, p^{N+1}, p^{N+2}, \ldots)$, $\sum |a_n x^n|$ is convergent.

$\square$

13.4 Definition of Radius of Convergence

Suppose $\{|x_0| : \sum a_n x_0^n \text{ convergent}\}$ is bounded; then let $R := \sup\{|x_0| : \sum a_n x_0^n \text{ convergent}\}$. If the set is not bounded, we write "$R = \infty$".
We call this the *Radius of Convergence.* Why? Well, see below.

Example. *With the same examples as above we get* $R = 1, \infty, 1, \infty, \infty$.

13.5 Why it is the 'radius'

Theorem. *Let* $\sum a_n x^n$ *have radius of convergence* R. *Then*

(i) $|x| < R$ *then* $\sum a_n x^n$ *is absolutely convergent*

(ii) $|x| > R$ *then* $\sum a_n x^n$ *is divergent*

(iii) $|x| = R$ *then all things are possible*

Proof.

(i) As $|x|$ is not sup, there exists an x_0 such that $|x| < x_0 < R$ and $\sum a_n x_0^n$ is convergent. Now use (13.3).

(ii) If $\sum a_n x^n$ convergent, we would have $R \geqslant |x|$ by definition of sup.

(iii)

Example.

$\sum x^n/n^2$	:	$R = 1$	:	*convergent at* $1, -1$
$\sum x^n/n$	:	$R = 1$	:	*convergent at* -1, *and divergent at* 1
$\sum (-x)^n/n$	:	$R = 1$	:	*convergent at* $+1$, *and divergent at* -1
$\sum x^n$	:	$R = 1$	:	*divergent at* $1, -1$

□

Note. *In the complex case 'all things are possible' is not so trivial.*

13.6 Examples of working out R

(i) Easy case: If $\lim a_{n+1}/a_n$ exists, and $= l$, then $R = 1/l$.

Proof. Put $u_n = |x^n a_n|$, $u_{n+1}/u_n = |x| a_{n+1}/a_n \to l|x|$. If $l|x| < 1$ then convergent, if $l|x| > 1$ then divergent. So by the previous theorem, $R = 1/l$. □

(ii) Comparison Test Case: $\sum x^{p_n}$, p =sequence of primes. By Comparison test, $\sum |x|^{p_n} \leqslant \sum |x|^n$ convergent if $|x| < 1$. But $\sum 1^{p_n}$ is divergent as there are infinitely many primes.

(iii) Cosine/sine etc.

Consider $\sum (-1)^n \frac{x^{2n}}{2n!}$: note $a_0 = 1$, $a_1 = 0$, $a_2 = 1/2$etc. So Ratio test is not good - yet!

Put $u_n := |(-1)^n x^{2n}/2n!|$. Then $u_{n+1}/u_n = \frac{x^2}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$. So convergent by Ratio Test!

So series absolutely convergent for all x, so convergent for all n. Therefore $R = \infty$.

13.7 Two Big Theorems

Theorem. *Suppose $\sum a_n x^n$ has radius of convergence R. Then*

(i) radius of convergence of $\sum (n+1) a_{n+1} x^n$ is R

(ii) radius of convergence of $\sum \frac{a_n}{n+1} x^n$ is R

and if $|x| < R$ then

$$\begin{aligned} \frac{d}{dx}\left(\sum a_n x^n\right) &= \sum (n+1) a_{n+1} x^n \\ \int_0^x \sum a_n t^n &= \sum \frac{a_n}{n+1} x^{n+1} \end{aligned}$$

Note. *There are definitions **not** content-free. We are **not** just differentiating/integrating a sum—but a limit!*

[*Clearly we can't prove these until we develop the theory of differentiation and integration in HT and TT.*]

14 The Elementary Functions

Again we will stick to the real case in this section.

14.1 The exponential function

(i) For all x, $\sum x^n/n!$ is convergent: so $R = \infty$.

(ii) Define

$$\exp(x) := \sum \frac{x^n}{n!}$$

(iii) $\exp(0) = 1$

(iv) $\exp(x+y) = \exp(x)\exp(y)$ [proved]

(v) $\exp(x) = \exp(x/2)^2 \geqslant 0$.

(vi) $\exp(x) > 0$ actually.
For $x > 0$,

$$\exp(x) \geqslant 1 + x > 0$$

for $x = -t, t > 0$,

$$\exp(x)\exp(-x) = \exp(0) = 1$$

So $\exp(-x) > 0$ too.

(vii) $x^\alpha \exp(-x) \to 0$ as $x \to \infty$. [*whatever this means...*]

Proof. Chose $x \in \mathbb{N}$, $n > \alpha$. Then for $x > 1$

$$\frac{\exp(x)}{x^\alpha} \geqslant \frac{\exp(x)}{x^n} \geqslant \frac{x^{n+1}/(n+1)!}{x^n} = \frac{x}{(n+1)!}$$

So

$$0 \leqslant x^\alpha \exp(-x) \leqslant \frac{(n+1)!}{x}$$

A sanwich argument completes the proof. □

(viii) For you: $\exp(x) = \exp(1)^x$.
[Write $x = \sup x_n$, $x_n \in \mathbb{Q}$, and prove that the result is true for each x_n—essentially algebra, done by MI. Passing to the limit is tougher.]

14.2 The Trigonometric Functions

(i) For all x

$$\sum \frac{x^{2n}(-1)^n}{2n!} \text{ and } \sum \frac{x^{2n+1}(-1)^{n-1}}{(2n+1)!}$$

are convergent.

(ii) Define

$$\cos x := \sum \frac{x^{2n}(-1)^n}{2n!}, \ \sin x := \sum \frac{x^{2n+1}(-1)^{n-1}}{(2n+1)!}$$

(iii)

$$\cos 0 = 1, \ \sin 0 = 0$$

(iv)

$$\cos x = \cos(-x), \ \sin x = -\sin(-x)$$

(v)

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

We identify coefficients of $\frac{x^r y^s}{(r+s)!}$ on both sides.

Ditto for $\cos(x+y)$.

(vi)

$$\cos^2 x + \sin^2 x = 1$$

Well, the first coefficient of each side is 1. Then, clearly only **even** coefficients can be non-zero. The coefficient of x^{2k} is

in $\cos^2 x$:

$$\begin{aligned} & \sum \frac{x^{2m}}{(2m)!}(-1)^m \frac{x^{2k-2m}}{(2k-2m)!}(-1)^{k-m} \\ = & \frac{1}{(2k)!}(-1)^k \sum_{m=0}^{k} \binom{2k}{2m} \\ = & \frac{1}{(2k)!}(-1)^k 2^{2k-1} \end{aligned}$$

in $\sin^2 x$:

$$\begin{aligned} & \sum \frac{1}{(2m+1)!}(-1)^m \frac{1}{(2k-2m-1)!}(-1)^{k-m-1} \\ = & (-1)\frac{(-1)^k}{(2k)!} \sum_{m=0}^{k-1} \binom{2k}{2m+1} \\ = & -(-1)\frac{(-1)^k}{(2k)!} 2^{2k-1}, \end{aligned}$$

So result will follow from what we've proved about multiplication of Absolutely convergent series.

(vii) Much better:

$$\begin{aligned} \cos'(x) &= -\sin x \\ \sin'(x) &= \cos x \end{aligned}$$

[*Depends of results we have not proved.*]

(viii) Define $\pi/2 := \inf\{x > 0 : \cos(x) = 0\}$

(ix) Now establish the periodicity:

$$\cos(x + 2\pi) = \cos(x) \quad \text{and} \quad \sin(x + 2\pi) = \sin(x)$$

14.3 Hyperbolic Functions

Define

$$\cosh x := \sum \frac{x^{2n}}{2n!},\ \sinh x := \sum \frac{x^{2n+1}}{(2n+1)!}$$

Then

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2},\ \sinh x = \frac{\exp(x) - \exp(-x)}{2}$$

and all follows.

14.4 The other trigonometric functions

Define

$$\begin{aligned} \sec(x) &:= \frac{1}{\cos x} \quad \text{when } x \neq 0 \ldots \\ \tan(x) &:= \frac{\sin x}{\cos x} \quad \ldots \end{aligned}$$

Next term!

14.5 Logarithm

Consider $\sum \frac{x^{n+1}}{n+1}$. The radius of convergence is 1. For $|x| < 1$ define

$$\log(1-x) := \sum \frac{x^{n+1}}{n+1}$$

14.6 Binomial Series

The series

$$\sum k = 0^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} x^k$$

has radius of convergence 1 (unless $\alpha \in \mathbb{N}$); its sum, when $|x| < 1$, is

$$(1+x)^{\alpha}.$$