Analysis I Sheet 2 MT05

Suprema and completeness

1 Determine whether each of the following sets S is (a) bounded above (b) bounded below. Where possible find $\sup(S)$ and $\inf(S)$, and decide whether $\max(S)$ and $\min(S)$ exist. Give proofs of your assertions.

(i) $S = \{2^n : n \in \mathbb{N}\}$; (*and, for more practice later,* $S = \{a^n : n \in \mathbb{N}\}$, *where* $a > 1$);
(ii) $S = \{(-1)^n + 1/n : n \in \mathbb{N}\}$;
(iii) $S = \{2^{-m} + 3^{-n} : m, n \in \mathbb{N}\}$.

[*In this question you may use only the axioms and properties of real numbers we have already established; in particular you may not use logarithms.*]

2 Prove that there is a unique real number a such that $a^3 = 2$.

3 Let S, T be non-empty bounded subsets of $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function; that is, $x < y \Rightarrow f(x) < f(y)$.

Prove that

(i) $S \cup T$ is bounded above and $\sup(S \cup T) = \max(\sup S, \sup T)$;
(ii) $S + T := \{s + t : s \in S, t \in T\}$ is bounded below and $\inf(S + T) = \inf S + \inf T$;
(iii) $f(S)$ is bounded above and $\sup(f(S)) = f(\sup(S))$;

Later, for more practice, you may like to use (iii), and what was proved in the lectures about $\sup\{-s | s \in S\}$, *to prove that if* $c < 0$ *then* $cS := \{cs : s \in S\}$ *is bounded above and* $\sup(cS) = c \inf S$.

4 Let $a, b \in \mathbb{R}$ with $a < b$. Prove that there is an irrational c such that $a < c < b$.

[*This can be done in a few lines using facts from lectures.*]

The complex numbers

5 (a) Prove that there does not exist a subset $\mathbb{P}$ of $\mathbb{C}$ satisfying the positivity axioms:

(P1) If $\alpha, \beta \in \mathbb{P}$ then $\alpha + \beta \in \mathbb{P}$;

(P2) If $\alpha, \beta \in \mathbb{P}$ then $\alpha\beta \in \mathbb{P}$;

(P3) For each $\alpha \in \mathbb{C}$, exactly one of the following holds: $\alpha \in \mathbb{P}$, $\alpha = 0$, $-\alpha \in \mathbb{P}$.

[*Apply the trichotomy axiom (P3) to* $0 + 1\mathrm{i}$ *(usually written* i*).*]

(b) Let z be the complex number with real part x and imaginary part y, so that $z = x + y\mathrm{i}$. Suppose that $y = 0$. Prove that $|z| = |x|$.

[*Before you start this part, refresh your memory by re-reading the definitions of the modulus of a complex number and the modulus of a real number.*]