Analysis I Sheet 3 **MT05**

Convergence of sequences

1 For each of the following a_n and for arbitrary $\varepsilon > 0$, find N such that $|a_n| < \varepsilon$ whenever $n \geqslant N$.

$$\text{(i)}\ \frac{1}{n^2} \quad \text{(ii)}\ \frac{1}{n(n-\pi)} \quad \text{(iii)}\ \mathrm{e}^{-n}\sin(n^2) \quad \text{(iv)}\ \frac{1}{\log(\log(n+2))} \quad \text{(v)}\ \sin\frac{1}{n}.$$

[*You may in this question (and others like it) use the "standard" properties of the functions* cos, sin, log, exp*; these properties will be proved during the year—without using these special examples.*]

2 Let $a_n = 1 + \dfrac{\mathrm{e}^{n/1000} - 1}{n^2}$. Evaluate (by calculator) $a_1, a_2, a_5, a_{10}, a_{101}, a_{1000}$. Does (a_n) converge? Repeat with $a_n = \dfrac{n^2 \cos n}{\mathrm{e}^{n/1000} - 1}$.

3 (a) Use the Binomial Theorem to show that $(1 + \frac{2}{\sqrt{n}})^n \geqslant n$ for all $n \geqslant 2$. Deduce that $n^{1/n} \to 1$ as $n \to \infty$.

(b) Show that $\dfrac{n!}{r!} \geqslant r^{n-r}$ if $n > r > 0$. Hence or otherwise, show that $(n!)^{1/n} \to \infty$ as $n \to \infty$.

(c)(i) Show that $\dfrac{n!}{(n-k)!} \geqslant \left(\dfrac{n}{2}\right)^k$ if $n \geqslant 2k$.

(ii) Let a be a real number with $a > 1$, and k be a positive integer. Prove that there exist an integer M and a positive real number c (both independent of n) such that $a^n \geqslant cn^{k+1}$ whenever $n > M$.

[*Write $a = 1 + b$ and use the Binomial Theorem.*.]

(iii) Deduce that $n^k/a^n \to 0$ as $n \to \infty$.

(iv) Deduce that $n^\alpha/a^n \to 0$ for every real number α.

4 (a) Let $\tau := \dfrac{1+\sqrt{5}}{2}$. Verify that $\tau > 1$ and that $\tau = 1 + \dfrac{1}{\tau}$.

(b) Let $a_1 = 1$, and suppose that for $n \geqslant 1$, $a_n > 0$ and $a_{n+1} = 1 + \dfrac{1}{a_n}$. Prove that $|a_{k+1} - \tau| = |a_k - \tau|/|a_k|\tau$, and (by induction on k) that $|a_{k+1} - \tau| \leqslant \left(\dfrac{1}{\tau}\right)^k$.

(c) Prove (from the definition) that $a_n \to \tau$ as $n \to \infty$.

The following question is ***optional****. It uses what we have proved about sequences to show that positive real numbers have* decimal *expressions; and points to one of the drawbacks of working with these expressions.*

5 Let $x > 0$.

(a) For each natural number $n \geqslant 1$, prove that there is a largest member of $\{k : k \in \mathbb{N} \cup \{0\}, k < 10^n x\}$. Denote this number by $D_n(x)$, and let $d_n(x) := D_n(x)/10^n$, which we will call the *n-th decimal truncation of x*.

(b) Find $d_n(1)$, $d_n(\frac{1}{9})$ and $d_n(\frac{5}{37})$. (Justify your answers).

(c) Prove that for all n, $x_n := 10^{n+1}(d_{n+1}(x) - d_n(x)) \in \{0,1,2,3,4,5,6,7,8,9\}$. We call this the n-th decimal digit of x.

(d) Prove that for every n there exists an $N > n$ such that $d_n(x) < d_N(x)$.

(e) Prove that $d_n(x) \leqslant d_{n+1}(x) < x$, and deduce that $d_n(x) \to x$ as $n \to \infty$.

(f) (i) Show by an example that $d_n(x+y) = d_n(x) + d_n(y)$ is not always true.

(ii) Suppose that $N \geqslant n$. Find (rational) numbers $x_1 = x_2$, y_1, y_2 such that $d_N(x_1) = d_N(x_2)$, $d_N(y_1) = d_N(y_2)$, but $d_n(x_1 + y_1) \neq d_n(x_2 + y_2)$.

(iii) Deduce that there cannot exist functions $\alpha : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ and $\beta : \mathbb{N} \to \mathbb{N}$ such that, for all positive x and y, $d_n(x+y) = \alpha\left(d_{\beta(n)}(x), d_{\beta(n)}(y)\right)$.